DYNAMIC MODELLING OF TWO LINK FLEXIBLE MANIPULATOR USING LAGRANGIAN ASSUMED MODES METHOD

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Abstract-: In this paper, a mathematical model of two link flexible manipulator is presented. The model is prepared using Lagrangian-assumed modes method. The links are modelled as Euler-Bernoulli beams. A literature survey containing the work done by various authors in the area of flexible robotics is provided. A general expression for the frequency equation for the vibrating links is obtained which is found to be time-dependent. The effect of payload on the natural frequencies of the links is studied. Simulation results are obtained using both forward and inverse dynamics. The results thus obtained are compared with that found in the literature.

Keywords- Assumed Modes Method, Dynamic Modelling, Flexible Manipulator, Time-dependent Frequency

INTRODUCTION

The dynamic modelling of flexible manipulators becomes extremely complicated due to the elastic deformation of the links. In the present work, dynamic model of a two link flexible manipulator modelled as Euler-Bernoulli beam is obtained using the approach of Lagrangian-assumed modes method (AMM). A short but conclusive literature survey regarding the work done by various authors in the area of flexible robotics was done. Benosman and Vey [1] have given a review of various techniques for modelling and control of flexible multi-body systems. The initial work on flexible robots started with manipulators having single flexible link modelled as Euler-Bernoulli (EB) beam. Assumed modes method was used to solve for the elastic deformation of the link. Lagrangian approach is the most preferred one for developing the mathematical model of the flexible manipulator system. Book et al [2] presented the frequency domain model and the time domain model of the distributed flexibility system. They used the linear feedback schemes for vibration control.

The correct values of input torques for the flexible manipulator were found by Luh et al [3] using resolved-motion-rate-controls (feedback) method. Book and Majette [4], Judd and Falkenburg [5] and Kanoh et al [6] have focussed upon the design of controllers for the flexible manipulators besides dynamic modelling. Thus, it can be observed that during the early years between 1975 and 1986, a lot of work on flexible manipulators was carried out that involved dynamic modelling and also the design of controllers using conventional approach. Bakr and Shabana [7] developed a method for the dynamic analysis of geometrically nonlinear inertia-variant flexible systems using Lagrange's multipliers. Till now, the manipulators having only single flexible link were considered but Luca and Siciliano [8] derived a closed-form finite-dimensional dynamic model for planar multilink lightweight robots. Li and Sankar [9] developed systematic methods for efficient modelling and forward dynamics computation of flexible manipulators. DU et. al [10] focussed upon geometric non-linearity caused by large elastic deflections of a flexible EB beam. Theodore and Ghosal [11] performed dynamic modelling of a flexible EB link with prismatic joint. Yuksel and Aksoy [12] investigated the flexural vibrations of a flexible linear EB beam with different base excitations. Ata et. al [13] highlighted the significance of inverse dynamic analysis for flexible manipulator and the effect of boundary conditions on the elastic deflection and the corresponding torques provided by the actuators.

I. MATHEMATICAL MODELLING

Mishra et al [14] have given a mathematical model of single link flexible manipulator. Based on the same approach a mathematical model of two link flexible manipulator having two revolute joints is developed. The effect of payload is also considered. The flexible links are modelled as clamped-free Euler-Bernoulli beams having a mass at the tip which act as a payload. The equations of motion are obtained using Lagrangian assumed modes method (AMM). This dynamic modelling considers only the flexural modes of vibrations. The expressions of joint torques consisting of inertia matrix, centrifugal and Coriolis' torques matrix, gravity matrix, stiffness matrix and other miscellaneous terms are obtained for the both joints.

The following assumptions are made before modelling:

1. The flexible link is considered as a distributed mass system.
2. The links undergo small elastic deformations.
3. Euler-Bernoulli beam theory with fixed-free boundary conditions can describe the vibratory motion of the flexible link.

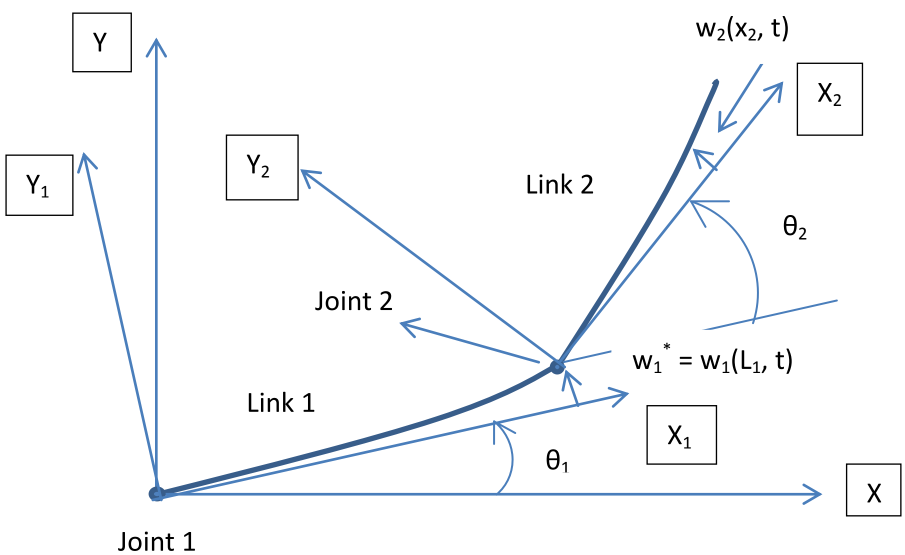

Figure1: Dynamic Modelling of a Two Link Flexible Manipulator Having Two Clamped-Free Euler-Bernoulli Beams and Two Revolute Joints

Figure 1 shows a two link flexible manipulator. There are three coordinate frames. The frame X-Y is the reference/ ground frame while the frames X_1-Y_1 and X_2-Y_2 are the local frames attached to link 1 and link 2 respectively. The axis X_1 is parallel to undeformed beam axis corresponding to link 1 and the axis X_2 is parallel to undeformed beam axis corresponding to link 2. The joint 1 is given a rigid rotation of θ_1 and the joint 2 is given a rigid rotation of θ_2.

The position of any point on link one with respect to ground is given by:

$$\{p_1\} = [T_1]\{r_1\} \qquad (1)$$

The position of any point on link two with respect to ground is given by:

$$\{p_2\} = [T_1]\{r_1^*\} + [T_1][T_2]\{r_2\} \qquad (2)$$

where,

$T_1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}$; $T_2 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}$; $\{r_1\} = \begin{Bmatrix} x_1 \\ w_1(x_1, t) \end{Bmatrix}$; $\{r_1^*\} = \begin{Bmatrix} L_1 \\ w_1(L_1, t) \end{Bmatrix}$
$\{r_2\} = \begin{Bmatrix} x_2 \\ w_2(x_2, t) \end{Bmatrix}$

L_1 and L_2 = length of link one and two respectively,
θ_1 and θ_2 = joint rotations (rigid) of joints one and two respectively,
x_1 and x_2 = distances measured along undeformed beam one and beam two axes, i.e. X_1 and X_2 respectively,
$w_1(x_1, t)$ and $w_2(x_2, t)$ = elastic displacements of links one and two respectively undergoing flexural vibrations, obtained by the solution of clamped-free Euler-Bernoulli beam equation
$\{r_1\}$ = position coordinates of any point on link one w.r.t to undeformed beam one axes i.e., X_1, Y_1
$\{r_2\}$ = position coordinates of any point on link two w.r.t to undeformed beam two axis i.e., X_2, Y_2
$\{r_1^*\}$ = position coordinates of the end point of link one w.r.t. undeformed beam one axis X_1
The velocity of any point on link one with respect to ground is given by:

$$\{\dot{p}_1\} = [T_1]\{\dot{r}_1\} + [\dot{T}_1]\{r_1\} \qquad (3)$$

The velocity of any point on link two with respect to ground is given by:

$$\{\dot{p}_2\} = [T_1]\{\dot{r}_1^*\} + [\dot{T}_1]\{r_1^*\} + [T_1][T_2]\{\dot{r}_2\} + [T_1][\dot{T}_2]\{r_2\} + [\dot{T}_1][T_2]\{r_2\} \qquad (4)$$

Now, total kinetic energy of the two link manipulator system is given by:
Total kinetic energy, K.E. = Kinetic energy of link one, K.E.$_1$ + Kinetic energy of link two, K.E.$_2$

Thus, $$K.E. = \frac{1}{2}\rho_1 A_1 \int_0^{L_1} \{\dot{p}_1\}' \{\dot{p}_1\} dx_1 + \frac{1}{2}\rho_2 A_2 \int_0^{L_2} \{\dot{p}_2\}' \{\dot{p}_2\} dx_2 \qquad (5)$$

Now, total potential energy of the two link flexible manipulator system is given by:
Total potential energy, P.E. = Potential energy of link 1, P.E.$_1$ + Potential energy of link 2, P.E.$_2$
The potential energy of each link is calculated by adding the strain energy and gravitational potential energy for that link. So, we get
Potential energy of link i (i = 1, 2) = Strain energy of link i, S.E.$_i$ + Gravitational potential energy of link i, G.P.E.$_i$
Hence, P.E. = (S.E.$_1$ + G.P.E.$_1$) + (S.E.$_2$ + G.P.E.$_2$)

Thus, $$P.E. = \frac{1}{2}E_1 I_1 \int_0^{L_1} \left(\frac{\partial^2 w_1}{\partial x_1^2}\right)^2 dx_1 + \rho_1 A_1 \int_0^{L_1} \{g\}' \{p_1\} dx_1 + \frac{1}{2}E_2 I_2 \int_0^{L_2} \left(\frac{\partial^2 w_2}{\partial x_2^2}\right)^2 dx_2 + \rho_2 A_2 \int_0^{L_2} \{g\}' \{p_2\} dx_2 \qquad (6)$$

where, ρ_i = density of link i; A_i = area of cross-section of link i ; E_i = Young's modulus of elasticity of link i, I_i = area moment of inertia of link i; i = (1,2): denotes link number; the superscript- (′) denotes transpose of a matrix; w_1 and w_2 => $w_1(x_1, t)$ and $w_2(x_2, t)$ respectively; $\{g\}' = \{0 \quad g\}$
Now, Lagrangian is defined as the difference of kinetic energy and potential energy. Therefore, Lagrangian of the two link flexible manipulator system is given by:
Lagrangian, L = K.E. – P.E.
Using dynamics equations we can now obtain the values of joint torques. The dynamics equation is given as follows:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j \qquad (7)$$

where, q_j represents the generalized coordinate associated with joint j and Q_j represents the generalized external force/ torque applied at joint j ; j represent the joint number (for the present case, j = 1 and 2)

After using the dynamics equation, we obtain the equation of motion of the two link flexible manipulator system in the matrix form as follows:

$$[M(t)]_{5X5}\{\ddot{q}\}_{5X1} + [H(t)]_{5X1} + [G(t)]_{5X1} + [C(t)]_{5X5}\{\dot{q}\}_{5X1} + [K(t)]_{5X5}\{q\}_{5X1} + [K(t)]^{\#}_{5X1} = \{Q(t)\}_{5X1} \qquad (8)$$

where, [M(t)] = inertia matrix, [H(t)] = centrifugal and Coriolis forces/ torques matrix; [G(t)] = gravity matrix; [C(t)] = matrix associated with velocity dependent terms; [K(t)] = stiffness matrix; $[K(t)]^{\#}$ = matrix containing miscellaneous terms; [Q(t)] = force/ torque matrix

$\{q\} = \{\theta_1 \quad \theta_2 \quad w_1 \quad w_2 \quad w_1^*\}'$; $\{Q\} = \{\tau_1 \quad \tau_2 \quad 0 \quad 0 \quad 0\}'$

where, w_1^* => $w_1(L_1, t)$ = function of time only; it is obtained by putting $x_1 = L_1$ in the term $w_1(x_1, t)$

The terms- $\{w_i^*\}, \{\dot{w}_i^*\}$ *and* $\{\ddot{w}_i^*\}$ represent the elastic motion (displacement, velocity and acceleration respectively) of the end point of link i.

A. Assumed Modes Method

It has been mentioned before that the term $w_i(x_i, t)$ for any link i can be found out by the solution of equation of motion of Euler-Bernoulli beam. Since, a flexible link is a continuous system, its solution is given as follows:

$$w_i(x_i, t) = \sum_{n=1}^{\infty} W_n(x_i)T_n(t) \qquad (9)$$

where, n = number of mode (n = 1, 2, ….∞); $W_n(x_i)$ = nth mode shape and is a function of distance x measured along undeformed beam axis for link i; $T_n(t)$ = time dependent function of nth mode

Since it is impossible to include all the infinite number of modes of the system, hence it is modelled with reduced number of modes by assuming some definite number of modes which best describe the behavior of the system. Thus, we can rewrite the above equation with reduced number of modes say m, as follows:

$$w_i(x_i, t) = \sum_{n=1}^{m} W_n(x_i)T_n(t) \qquad (9a)$$

where, m = number of assumed modes; it is a finite integer

For the present case, only first two modes are considered, i.e. m = 2.

The boundary conditions used for the two flexible links are as follows:

i) $w_i(0, t) = 0$ (9b)

ii) $\frac{\partial w_i(0,t)}{\partial x_i} = 0$ (9c)

iii) $E_i I_i \frac{\partial^2 w_i(L_i,t)}{\partial x_i^2} = -J_{pi}\left[\frac{\partial^2}{\partial t^2}\left(\frac{\partial w_i(L_i,t)}{\partial x_i}\right)\right]$ (9d)

iv) $E_i I_i \frac{\partial^3 w_i(L_i,t)}{\partial x_i^3} = M_{pi}\left[\frac{\partial^2 w_i(L_i,t)}{\partial t^2}\right]$ (9e)

where, M_{pi} and J_{pi} = effective mass and effective mass moment of inertia respectively of the payload attached at the end point of link i; subscript p stands for payload.

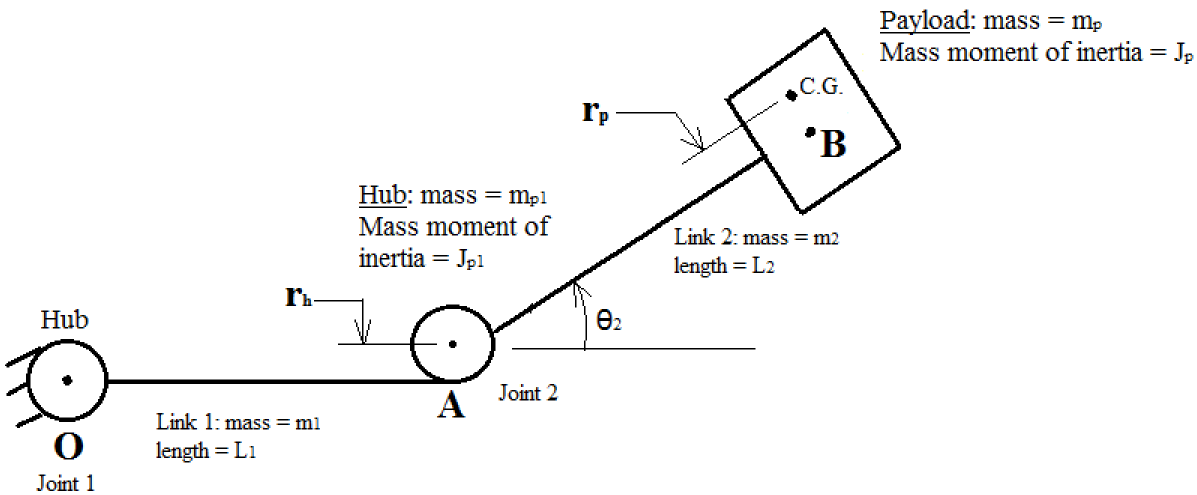

Figure 2: Calculation of Effective Inertia at the End of Links.

Referring to figure 2 where the joint 1 is made fixed, the effective mass and effective mass moment of inertia at the end of each link can be found out as follows:

For link 1, $M_p = m_{p1} + (m_2 + m_p)\cos^2\theta_2$ (10a)

and $J_p = J_{p1A} + J_{2A} + J_{pA}$ (10b)

For link 2, $M_p = m_p$ (10c)

and $J_p = J_{pCG} + m_p r_p^2$ (10d)

where, m_{p1} = mass of actuator placed at the end of link 1; m_2 = mass of link 2; m_p = mass of payload attached at the end of link2; J_{p1A} = mass moment of inertia of actuator placed at the end of link 1 about point A = $J_{p1} + m_{p1} r_{p1}^2$; (J_{p1} = mass moment of inertia of hub at joint 2 about its own center of gravity); J_{2A} = mass moment of inertia of link 2 about point A = $J_2 + m_2(L_2/2)^2$; (J_2 = mass moment of inertia of link 2 about its own center of gravity); J_{pA} = mass moment of inertia of payload attached at the end of link 2 about point A = $J_{pCG} + m_p r_p^2 + m_p L_2^2$; CG stands for center of gravity; r_{p1} = distance of center of gravity of hub at the end of link 1,i.e. at joint 2 from point A, measured perpendicular to the undeformed beam axis X_1; r_p = distance of CG of payload at the end of link 2 from point B, measured perpendicular to the undeformed beam axis X_2 .

Using above mentioned boundary conditions, the expression for mode shape can be found out as follows:

$$W_n(x_i, t) = C_{1n}[\cosh(\beta_n x_i) - \cos(\beta_n x_i) + \alpha_n\{\sin(\beta_n x_i) - \sinh(\beta_n x_i)\}] \quad (11)$$

where,

$$\alpha_n = \left(\frac{\sinh(\beta_n L_i) - \sin(\beta_n L_i) + a_i\beta_n L_i(\cosh(\beta_n L_i) - \cos(\beta_n L_i))}{\cos(\beta_n L_i) + \cosh(\beta_n L_i) + a_i\beta_n L_i(\sinh(\beta_n L_i) - \sin(\beta_n L_i))}\right);$$

$\beta_n^4 = \left(\frac{\rho_i A_i}{E_i I_i}\right)\omega_n^2$; ω_n = nth mode natural angular frequency of the beam i; C_{1n} = normalization constant

The frequency equation of any link i undergoing flexural vibrations is also derived and found to be varying with time. The frequency equation is as follows:

$$(1+\cosh(z_{ni})\cos(z_{ni})+ a_i z_{ni}(\cos(z_{ni})\sinh(z_{ni})-\cosh(z_{ni})\sin(z_{ni}))- b_i z_{ni}^3(\cosh(z_{ni})\sin(z_{ni})+\cos(z_{ni})\sinh(z_{ni}))- a_i b_i z_{ni}^4(\cosh(z_{ni})\cos(z_{ni})-1)=0 \quad (12)$$

where, $z_{ni}=\beta_{ni}L_i$; $a_i=\frac{M_{pi}}{\rho_i A_i L_i}$; $b_i=\frac{J_{pi}}{\rho_i A_i L_i^3}$

The roots of the frequency equation (eqn. 12) are given by the values of z_{ni}. The frequency equation is time-dependent and hence the natural frequencies are also time-dependent. Ghosal [15] has also discussed about the time-dependency of the frequency equation.

The time dependent term is given as follows:

$$T_n(t)=A_n\cos(\omega_n t)+B_n\sin(\omega_n t) \quad (13a)$$

If damping is present then equation 13a gets modified to equation 13b.

$$T_n(t)=e^{-\xi_n\omega_n t}\{A_n\cos(\omega_{dn}t)+B_n\sin(\omega_{dn}t)\} \quad (13b)$$

where, ξ_n = nth mode damping ratio,

ω_n = nth mode natural angular frequency,

ω_{dn} = nth mode damped angular frequency = $\omega_n\sqrt{1-\xi_n^2}$

Assuming the following initial conditions:

i) $w(x_i,0)=w_0$, (13c)

ii) $\dot{w}(x_i,0)=0$, (13d)

and the validity of the principal of orthogonality of normal modes of vibration, we obtain the following values of A_n and B_n:

$$A_{ni}=\frac{\int_0^{L_i}(w_0 W_n(x))dx_i}{\int_0^{L_i}[W_n(x)]^2 dx_i} \; ; \quad (13e)$$

$$B_{ni}=0 \quad (13f)$$

B. Effect of Payload Inertia and Hub Inertia

If the mass and mass moment of inertia of the two hubs located at the two joints (refer Fig. 2) are considered, the expressions of total kinetic energy and total potential energy gets altered by the addition of following terms:

$$K.E._h=\frac{1}{2}I_{hO}\dot{\theta}_1^2+\frac{1}{2}I_h\left(\dot{\theta}_2\right)^2+\frac{1}{2}m_h\{v_{mh}\}'\{v_{mh}\} \quad (14)$$

$$P.E._h=m_h\{g\}'\{p_{mh}\} \quad (15)$$

$$\{p_{mh}\}=[T_1][\{r_1^*\}+\{e_1\}] \quad (16)$$

$$\{v_{mh}\}=\{\dot{p}_1^*\}+[\dot{T}_1]\{e_1\} \quad (17)$$

where, I_{hO} and I_h = mass moment of inertias of hubs situated at joint 1 and joint 2 respectively; m_h = mass of hub at joint 2; p_{mh} and v_{mh} = position and velocity coordinates respectively of the hub situated at joint 2 measured w.r.t. local frame X_1-Y_1; $\{e_1\}=\{e_{1x} \quad e_{1y}\}'$ is the location of center of gravity of hub at joint 2 measured w.r.t. local frame X_1-Y_1; subscript h stands for hub. Due to the addition of these extra terms, the Lagrangian of the system gets altered and thus the joints torques also get changed. Similarly, if a payload (refer to Fig. 2) is attached to the flexible manipulator system, the equation of motion of the system gets altered as follows:

$$[M_{mod}(t)]_{6X6}\{\ddot{q}_{mod}\}_{6X1}+[H_{mod}(t)]_{6X1}+[G_{mod}(t)]_{6X1}+[C_{mod}(t)]_{6X6}\{\dot{q}_{mod}\}_{6X1}+[K_{mod}(t)]_{6X6}\{q_{mod}\}_{6X1}+[K_{mod}(t)]^{\#}{}_{6X1}=\{Q_{mod}(t)\}_{6X1} \quad (18)$$

where the symbols have their usual meanings; the subscript 'mod' stands for modified and
$\{\ddot{q}_{mod}\} = \{\ddot{\theta}_1 \quad \ddot{\theta}_2 \quad \ddot{w}_1 \quad \ddot{w}_2 \quad \ddot{w}_1^* \quad \ddot{w}_2^*\}'$; $\{\dot{q}_{mod}\} = \{\dot{\theta}_1 \quad \dot{\theta}_2 \quad \dot{w}_1 \quad \dot{w}_2 \quad \dot{w}_1^* \quad \dot{w}_2^*\}'$;
$\{q_{mod}\} = \{\theta_1 \quad \theta_2 \quad w_1 \quad w_2 \quad w_1^* \quad w_2^*\}'$;
$\{Q_{mod}\} = \{\tau_{1mod} \quad \tau_{2mod} \quad 0 \quad 0 \quad 0 \quad 0\}'$
The alteration in the equation of motion is due to the inclusion of a new term- w_2^* and its derivatives, which represent the elastic motion of the end point of link 2. The term w_2^* is obtained by putting $x_2 = L_2$ in the term $w_2(x_2, t)$, i.e. $w_2(L_2, t)$. Furthermore, the matrices- M, H, G, C, K and $K^{\#}$ also get modified due to the presence of payload.
The general expression for the joint torque is given by equation 19 given below.

$$\tau_j(t) = M_{j1}\ddot{\theta}_1 + M_{j2}\ddot{\theta}_2 + M_{j3}\ddot{w}_1 + M_{j4}\ddot{w}_2 + M_{j5}\ddot{w}_1^* + H_j + G_j + C_{j1}\dot{\theta}_1 + C_{j2}\dot{\theta}_2 + C_{j5}\dot{w}_1^* + K_{j5}w_1^* + K_j^{\#} \quad (19)$$

where, subscript 'j' denotes joint number (j = 1,2). The coefficients representing the elements in the first two rows of the inertia matrix $[M_j(t)]_{5X5}$ are evaluated using the expressions provided in equation 20.

$$M_{11} = \frac{1}{2}\rho_1 A_1\left[\frac{2}{3}L_1^3 + \int_0^{L_1} w_1^2\,dx_1\right] + \frac{1}{2}\rho_2 A_2\left[2L_2(L_1^2 + w_1^{*2}) + \frac{2}{3}L_2^3 + 2\int_0^{L_2} w_2^2\,dx_2 + 2L_2^2(L_1\cos\theta_2 + w_1^*\sin\theta_2) + 4(w_1^*\cos\theta_2 - L_1\sin\theta_2)\int_0^{L_2} w_2\,dx_2\right];$$

$$M_{12} = \frac{1}{2}\rho_2 A_2\left[L_2^2(L_1\cos\theta_2 + w_1^*\sin\theta_2) + \frac{2}{3}L_2^3 + 2\int_0^{L_2} w_2^2\,dx_2 + 2(w_1^*\cos\theta_2 - L_1\sin\theta_2)\int_0^{L_2} w_2\,dx_2\right];$$

$$M_{13} = \frac{1}{2}\rho_1 A_1 L_1^2;$$

$$M_{14} = \frac{1}{2}\rho_1 A_1 L_2^2;$$

$$M_{15} = \frac{1}{2}\rho_2 A_2\left[2L_1L_2 + L_2^2\cos\theta_2 - 2\sin\theta_2\int_0^{L_2} w_2\,dx_2\right];$$

$$M_{21} = M_{12};$$

$$M_{22} = \frac{1}{2}\rho_2 A_2\left(\frac{2}{3}L_2^3 + 2\int_0^{L_2} w_2^2\,dx_2\right);$$

$$M_{23} = 0;$$

$$M_{24} = \left(\frac{-1}{2}\right)\rho_2 A_2 L_2^2;$$

$$M_{25} = \frac{1}{2}\rho_2 A_2\left[-L_2^2\cos\theta_2 + 2\sin\theta_2\int_0^{L_2} w_2\,dx_2\right]; \quad (20)$$

The coefficients representing the elements in the first two rows of the centrifugal/Coriolis torque matrix $[H_j(t)]_{5X1}$ are evaluated using the expressions given in equation (21).

$$H_1 = \frac{1}{2}\rho_2 A_2 \Big[\dot\theta_1\dot\theta_2 \Big\{2L_2^2(w_1^* \cos\theta_2 - L_1 \sin\theta_2) - 4(w_1^* \sin\theta_2 + L_1 \cos\theta_2)\int_0^{L_2} w_2 dx_2\Big\} + \dot\theta_2^2\Big\{L_2^2(-L_1 \sin\theta_2 + w_1^* \cos\theta_2) - 2(L_1 \cos\theta_2 + w_1^* \sin\theta_2)\int_0^{L_2} w_2 dx_2\Big\} + \dot w_1^*\dot\theta_1\Big\{4L_2 w_1^* + 4\cos\theta_2 \int_0^{L_2} w_2 dx_2 + 2L_2^2 \sin\theta_2\Big\}\Big] ;$$

$$H_2 = \frac{1}{2}\rho_2 A_2 \Big[\dot\theta_1^2\Big\{L_2^2(L_1 \sin\theta_2 - w_1^* \cos\theta_2) + 2(w_1^* \sin\theta_2 + L_1 \cos\theta_2)\int_0^{L_2} w_2 dx_2\Big\} + \dot w_1^*\dot\theta_1\Big\{2L_2^2 \sin\theta_2 + 4\cos\theta_2 \int_0^{L_2} w_2 dx_2\Big\}\Big] ;$$

(21)

The coefficients representing the elements in the first two rows of the gravity matrix $[G_j(t)]_{5X1}$ are evaluated using the expressions given in equation 22.

$$G_1 = g\Big[\rho_1 A_1\Big\{\frac{1}{2}L_1^2 \cos\theta_1 - \sin\theta_1 \int_0^{L_1} w_1 dx_1\Big\} + \rho_2 A_2\Big\{L_1 L_2 \cos\theta_2 - w_1^* L_2 \sin\theta_1 + \frac{1}{2}L_2^2 \cos(\theta_1 + \theta_2) - \sin(\theta_1 + \theta_2)\int_0^{L_2} w_2 dx_2\Big\}\Big] ;$$

$$G_2 = g\Big[\rho_2 A_2\Big\{\frac{1}{2}L_2^2 \cos(\theta_1 + \theta_2) - \sin(\theta_1 + \theta_2)\int_0^{L_2} w_2 dx_2\Big\}\Big] ;$$

(22)

The coefficients representing the elements in the first two rows of the velocity dependent/ gyroscopic matrix $[C_j(t)]_{5X5}$ are evaluated using the expressions provided by equation 23.

$$C_{11} = \frac{1}{2}\rho_1 A_1\Big[2\frac{d}{dt}\Big(\int_0^{L_1} w_1^2 dx_1\Big)\Big] - \frac{1}{2}\rho_2 A_2\Big[4(w_1^* \cos\theta_2 - L_1 \sin\theta_2)\frac{d}{dt}\Big(\int_0^{L_2} w_2 dx_2\Big)\Big] ;$$

$$C_{12} = \Big(\frac{-1}{2}\Big)\rho_2 A_2\Big[2(w_1^* \cos\theta_2 - L_1 \sin\theta_2)\int_0^{L_2} \dot w_2 dx_2 + 2(w_1^* \cos\theta_2 - L_1 \sin\theta_2)\frac{d}{dt}\int_0^{L_2} w_2 dx_2 + 2\frac{d}{dt}\int_0^{L_2} w_2^2 dx_2\Big] ;$$

$C_{13} = C_{14} = 0;$

$$C_{15} = \Big(\frac{-1}{2}\Big)\rho_2 A_2\Big[2\sin\theta_2\Big\{-\frac{d}{dt}\Big(\int_0^{L_2} w_2 dx_2\Big) + \int_0^{L_2} \dot w_2 dx_2\Big\}\Big] ;$$

$$C_{21} = \Big(\frac{1}{2}\Big)\rho_2 A_2\Big[2(w_1^* \cos\theta_2 - L_1 \sin\theta_2)\frac{d}{dt}\int_0^{L_2} w_2 dx_2 + 2(-w_1^* \cos\theta_2 + L_1 \sin\theta_2)\int_0^{L_2} \dot w_2 dx_2 + 2\frac{d}{dt}\int_0^{L_2} w_2^2 dx_2\Big] ;$$

$C_{23} = C_{24} = 0;$

$C_{25} = \frac{1}{2}\rho_2 A_2 \left[2\sin\theta_2 \frac{d}{dt}\left(\int_0^{L_2} w_2\, dx_2\right) + 2\sin\theta_2 \int_0^{L_2} \dot{w}_2\, dx_2\right]$; (23)

The coefficients representing the elements in the first two rows of the stiffness matrix $[K_j(t)]_{5X5}$ are evaluated using the expressions given in equation 24.

$K_{11} = K_{12} = K_{13} = K_{14} = K_{21} = K_{22} = K_{23} = K_{24} = K_{25} = 0$;

$K_{15} = \left(\frac{-1}{2}\right)\rho_2 A_2 \left[2\sin\theta_2 \frac{d}{dt}\left(\int_0^{L_2} \dot{w}_2\, dx_2\right)\right]$; (24)

The coefficients representing the elements in the first two rows of the miscellaneous matrix $[K_j^{\#}(t)]_{5X1}$ are evaluated using the expressions given in equation 25.

$K_1^{\#} = \left(\frac{-1}{2}\right)\rho_1 A_1 \left[\frac{d}{dt}\left(\int_0^{L_1} x_1^2 \left(\frac{dw_1}{dx_1}\right) dx_1\right)\right] - \frac{1}{2}\rho_2 A_2 \left[2L_1 \frac{d}{dt}\left(\int_0^{L_2} \dot{w}_2\, dx_2\right)\cos\theta_2 + 2\frac{d}{dt}\left(\int_0^{L_2} w_2^2\, dx_2\right) - \frac{d}{dt}\left(\int_0^{L_2} x_2^2 \left(\frac{dw_2}{dx_2}\right) dx_2\right)\right]$;

$K_2^{\#} = \frac{1}{2}\rho_2 A_2 \left[\frac{d}{dt}\left(\int_0^{L_2} x_2^2 \left(\frac{dw_2}{dx_2}\right) dx_2\right)\right]$; (25)

Thus, from the expressions of joint torque for the both joints it can be found out that these expressions are highly non-linear and involve strong coupling between the joint variables. An explicit solution cannot be obtained for these expressions. In order to solve these equations, numerical methods have to be adopted.

II. RESULTS

The simulation results are obtained using inverse dynamics. Table 1 shows the link parameters for the flexible manipulator.

A sinusoidal input of amplitude 1 rad and time period 1 sec is applied at joint one while a sinusoidal input of amplitude 1 rad and time period 0.5 sec is applied at the joint 2.

Table 1: Link Parameters for Two Link Flexible Manipulator

Link parameters	**Value**
Length of each link, $L_1 = L_2$	0.5 m
Density of material of first link, ρ_1	7850 kg/m^3
Density of material of second link, ρ_2	7850 kg/m^3
Area of cross-section of first link, A_1	7.9423 X 10^{-6} m^2
Area of cross-section of second link, A_2	7.9423 X 10^{-6} m^2
Area moment of inertia of first link, I_1	5 X 10^{-12} m^4
Area moment of inertia of second link, I_2	5 X 10^{-12} m^4
Hub inertia of joint 1, J_{h1}	0.02 kg-m^2
Hub inertia of joint 2, J_{h2}	0.02 kg-m^2
Modal damping ratios (link 1)	0.005 for all three modes
Modal damping ratios (link 2)	0.08 for all three modes

Table 2 shows the time-dependent behavior of natural frequency of the first link of the flexible manipulator.

The mode shapes of link 1 and link 2 at no payload are shown in figure 3.

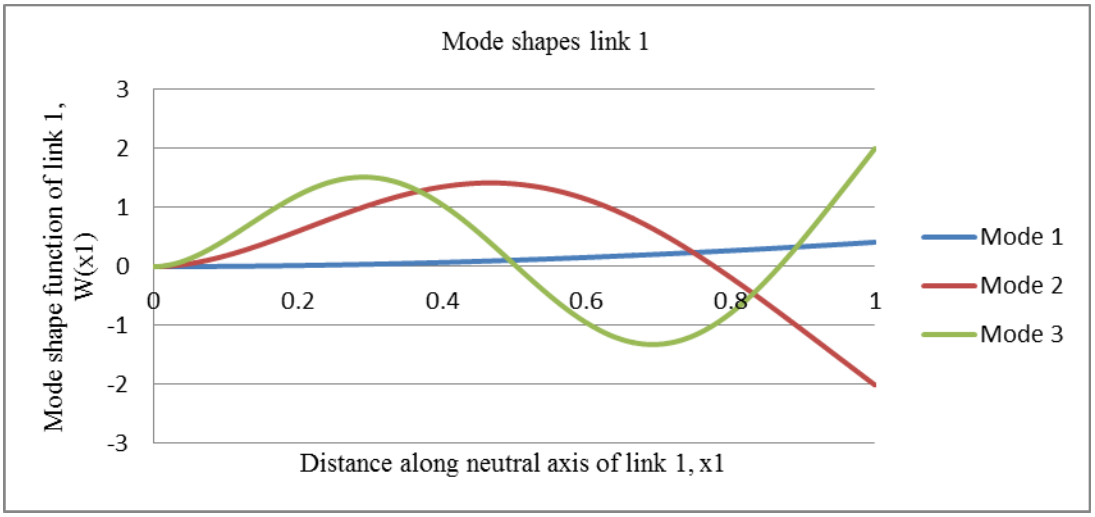

Figure 3a: Mode Shapes of Link One

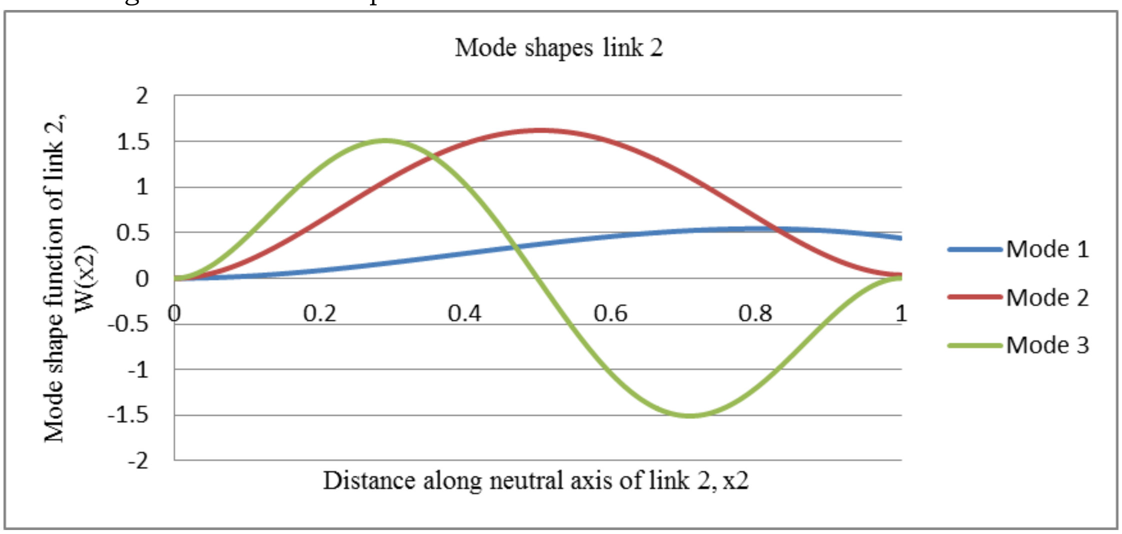

Figure 3b: Mode Shapes of Link Two

Figure 3: Mode Shapes of Link 1 and Link 2 of the Two Link Flexible Manipulator

From figures 3a and 3b it can be observed that the mode shapes conform to our assumption of fixed-free beam.

Table 2: Variation of Natural Frequencies of Links of the Two Link Flexible Manipulator at Different Payloads Attached at the Tip of the Second Link for Joint Inputs in the Form of Sinusoidal Functions

Payload (kg)	**Link 1**						**Link 2**		
	Mode 1 (Hz)		Mode 2 (Hz)		Mode 3 (Hz)		Mode 1 (Hz)	Mode 2 (Hz)	Mode 3 (Hz)
	Min	Max	Min	Max	Min	Max			
0	3.278	3.284	57.269	57.272	157.42	157.43	8.96	56.18	157.3
0.1	2.446	2.479	57.215	57.227	157.39	157.41	2.38	40.03	128.2
0.2	2.104	2.167	57.191	57.21	157.38	157.4	1.71	39.68	127.8

From table 2 it can be observed that the natural frequencies of links decrease with increase in the mass of payload attached at the tip of the second link. Besides this, link 1 exhibits natural frequencies having minimum and maximum values at a given value of payload. This is due to the time-dependency of natural frequencies of the first link. It is also observed that the natural frequency in the fundamental mode is more prone to any variation in payload than the other two modes. Since the payload is attached at the tip of the second link, so the relative change in the natural frequencies of the second link is more than that of the first link.

A. Comparison of results

This section presents the comparison of results for the flexible manipulator. The results are compared with the simulation results of Mohan [16]. Mohan has used forward dynamics to obtain the results and hence, forward dynamics is used to obtain the simulation results in the present work. The effect of gravity is not considered. For the two link flexible manipulator, joint 1 is provided a torque of 0.011 Nm while joint 2 is given a torque of 0.007 Nm. Table 3 shows the link parameters for two link flexible manipulator. Mohan [16] has considered first two modes of vibration only. He has used Newton-Euler's approach for modeling the flexible manipulator system. He has presented a recursive algorithm to solve the differential equations using Decoupled Natural Orthogonal Complement matrices. In the present work, first two modes of vibration are used. The differential equations of motion are solved using ode45 solver.

Table 3: Link Parameters for a Flexible Manipulator with Two Links [16]

	Link parameter	**Value**
Link 1	Length	0.24 m
	Area of cross-section	1.2 X 10^{-5} m^2
	Link mass	0.020 kg
	Mass per unit length	0.0833 kg/m
	Flexural rigidity	0.05 Nm^2
	Hub inertia	3 X 10^{-5}
	Mode 1 damping ratio	0.2
	Mode 2 damping ratio	0.04
	Payload	0.19 kg
Link 2	Length	0.19 m
	Area of cross-section	1.2 X 10^{-5} m^2
	Link mass	0.017 kg
	Mass per unit length	0.1053 kg
	Flexural rigidity	0.05
	Mode 1 damping ratio	0.2
	Mode 2 damping ratio	0.04
	Payload	0 kg

Fig.4 and Fig. 5 compare the joint responses obtained by present work and Mohan [16].

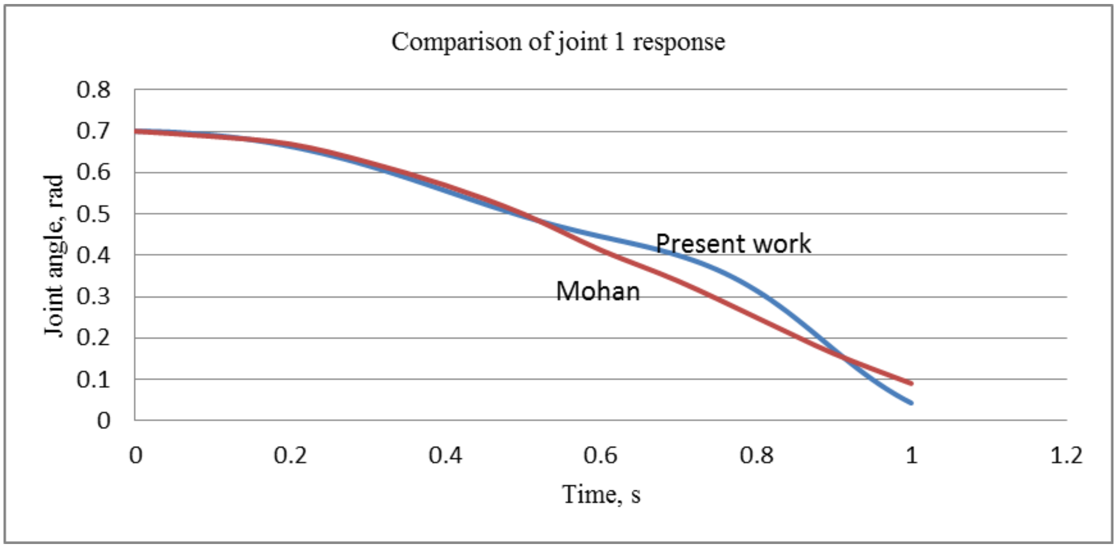

Fig.4: Comparison of Joint 1 Responses of Present Work and Mohan's [16] Work

From Fig. 4, it can be observed that the two curves deviate after about 0.5 sec. At 0.7 sec, the response of present work deviates from that of Mohan's by about 18%.

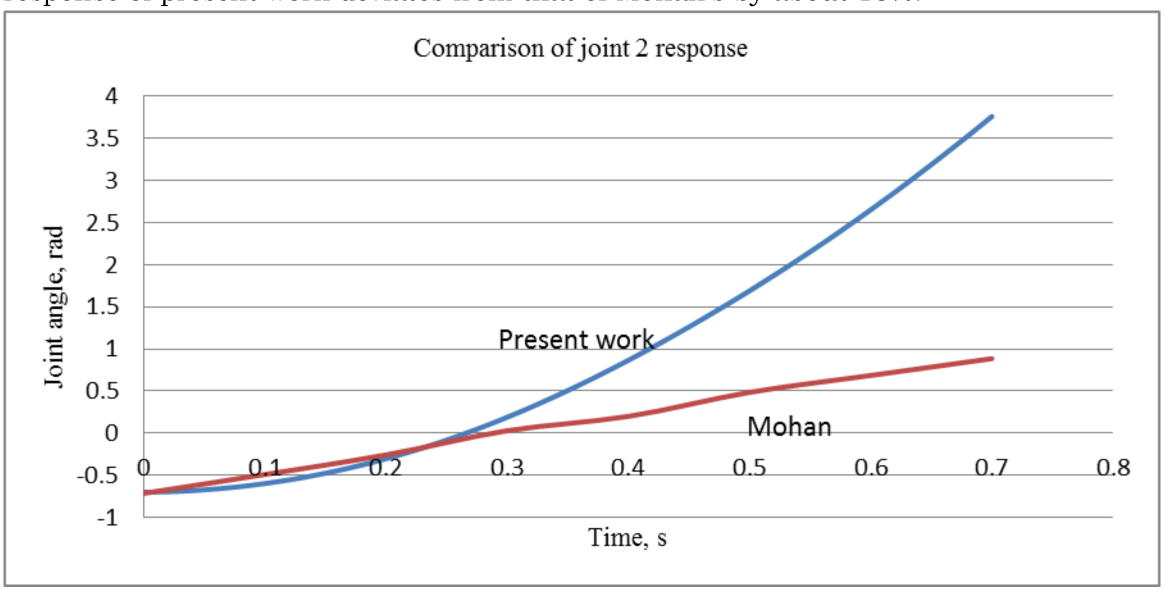

Fig.5: Comparison of Joint 2 Responses of Present Work and Mohan's [16] Work

From Fig. 5 it can be observed that the two responses match well up to 0.25 sec. After that the response of present work increases rapidly.

III. Conclusions

Most of the papers deal with the planar single link flexible robotic arms with small 3D motions. For such links, a linear model is sufficient to describe the dynamic characteristics. A lot of research is going on, on non-linear models of flexible arms. The simple case of non-linearity in flexible arms is that of a two link case which has been described in few papers. Furthermore, links having revolute joints have been studied. In this paper, a dynamic model of two link flexible manipulator having two revolute joints is obtained using Lagrangian-assumed modes method. The links are modelled as Euler-Bernoulli beams with fixed-free boundary conditions. The frequency equations for the both links are derived and it is found that link 1 exhibits time-dependent natural frequencies. The paper also compares the results of the present work with the work of

Mohan. Same types of assumptions are used. While Newton-Euler-AMM is followed by Mohan, Lagrangian-AMM is followed in present work. It is found that the response of joint 1 match up to certain extent with that of Mohan's but response of joint 2 does not match. The natural frequency of link 1 is found to be time-dependent.

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