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Einstein generalized Kropina conformal change of m-th root metrics

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Abstract: In this paper, we consider generalized Kropina conformal change of m-th root metric and prove that if it is an Einstein metric, then it is Ricci-flat. Moreover, if generalized Kropina conformal change of m-th root metric is weak Einstein metric, then it is also Ricci-flat. If generalized Kropina conformal change of m-th root metric is of scalar flag curvature $K(x, y)$ and isotropic S-curvature, then $K = 0$.

Keywords: Finsler space, generalized Kropina metrics, conformal change, m-th root metrics, Einstein metrics.
Mathematics Subject Classification: 53B40, 53C60.

1 Introduction

The conformal theory based on the theory of Finsler spaces by Matsumoto [9] has been developed by M. Hashiguchi [7]. Let F and $\bar{F}$ be two Finsler metrics on a manifold M such that $\bar{F} = e^{\sigma(x)}F$, where σ is a scalar function on M, then we call such two metrics F and $\bar{F}$ are conformally related. The conformal change is said to be a homothety if σ is a constant.

Recent studies show that m-th root Finsler metrics plays a very important role in physics, space-time and general relativity as well as in unified gauge field theory ([2], [4]). In 1979, H. Shimada [12] developed the theory of m-th root Finsler metrics, applied to ecology by Antonelli [1] and studied by several authors ([3], [12], [14], [17]). It is regarded as a generalization of Riemannian metric in the sense that for $m = 2$, 3 and 4, it is called Riemannian metric, cubic metric and quartic metric, respectively. Z. Shen and B. Li have studied the geometric properties of locally projectively flat fourth root metrics in the form $F = \sqrt[4]{a_{ijkl}(x)y^i y^j y^k y^l}$ and generalized fourth root metrics in the form $F = \sqrt{\sqrt{a_{ijkl}(x)y^i y^j y^k y^l} + b_{ij}(x)y^i y^j}$ [8]. A. Tayebi, T. Tabatabaeifar and E. Peyghan [13] introduced Kropina change of m-th root metric and established conditions on Kropina change of m-th root metric, to be locally dually flat and locally projectively flat.
Recently, B. Tiwari and M. Kumar [16] have studied Randers change of a Finsler space with m-th root metric. B. Tiwari and G. K. Prajapati [15] have also studied on Einstein Generalized Kropina change of m-th root Finsler metrics.

Let $(M, F) = F^n$ be an n-dimensional Finsler manifold. For a non-zero 1-form $\beta(x, y) = b_i(x)y^i$ on M, define a Finsler change as follows

$$F(x, y) \to \bar{F}(x, y) = f(F, \beta),$$

where $f(F, \beta)$ is a positively homogeneous function of F and β.

A Finsler change is called Kropina change if $f(F, \beta) = \frac{F^2}{\beta}$ and generalized Kropina change if $f(F, \beta) = \frac{F^{k+1}}{\beta^k}$, where k is any positive integer.

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The purpose of this paper is to study generalized Kropina conformal change of m-th root metrics, defined by

$$\bar{F} = e^{\sigma(x)} \frac{F^{k+1}}{\beta^k}, \tag{1.1}$$

where $F = \sqrt[m]{A}$ is an m-th root metric, for which we shall restrict our consideration on β, where $\beta > 0$.

In General Relativity, the Einstein metrics are solutions to Einstein field equation, which closely connect Riemannian geometry with gravitation. C. Robles studied a special class of Einstein Finsler metrics, that is, Einstein Randers metrics, and proved that for a Randers metric on a 3-dimensional manifold, it is Einstein if and only if it has constant flag curvature. E. Guo, X. Mo and X. Zhang have explicitly constructed an Einstein Finsler metrics of non-constant flag curvature in terms of navigation representation [6]. Recently, Z. Shen and C. Yu, using certain transformation, construct a large class of Einstein metrics [10]. In this paper, we establish following theorems

Theorem 1.1 *Let $\bar{F} = e^{\sigma(x)} \frac{F^{k+1}}{\beta^k}$ be a non-Riemannian generalized Kropina conformal change of m-th root metric F on a manifold of dimension $n \geq 2$, with $m \geq 3$ and $m \nmid (k+1)$. If $\bar{F}$ is Einstein metric, Then it is Ricci-flat.*

Theorem 1.2 *Let $\bar{F} = e^{\sigma(x)} \frac{F^{k+1}}{\beta^k}$ be a non-Riemannian generalized Kropina conformal change of m-th root metric F on a manifold of dimension $n \geq 2$, with $m \geq 3$ and $m \nmid (k+1)$. If $\bar{F}$ is a weak Einstein metric, Then it is Ricci-flat.*

Theorem 1.3 *Let $\bar{F} = e^{\sigma(x)} \frac{F^{k+1}}{\beta^k}$ be a non-Riemannian generalized Kropina conformal change of m-th root metric F on a manifold of dimension $n \geq 2$, with $m \geq 3$ and $m \nmid (k+1)$. If $\bar{F}$ is of scalar flag curvature $K(x,y)$ with isotropic S-curvature, then $K = 0$.*

Throughout the paper we call the Finsler metric $\bar{F}$ as generalized Kropina conformal change of m-th root metric and $\bar{F}^n = (M, \bar{F})$ as generalized Kropina conformal transformed Finsler space. We restrict ourselves for $m \geq 3$, throughout the paper and also the quantities corresponding to the generalized Kropina conformal transformed Finsler space $\bar{F}^n$ will be denoted by putting bar on the top of that quantity.

2 Preliminaries

Let M be an n-dimensional C^∞-manifold. Denote T_xM, the tangent space of M at x. The tangent bundle TM is the union of tangent spaces, $TM := \bigcup_{x\in M} T_xM$. We denote the elements of TM by (x,y), where $x = (x^i)$ be a point of M and $y \in T_xM$ called supporting element. We denote $TM_0 = TM \setminus \{0\}$.

Definition: A Finsler metric on M is a function $F : TM \to [0,\infty)$ with the following properties:
(i) F is C^∞ on TM_0,
(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM and
(iii) the Hessian of $\frac{F^2}{2}$ with element $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is positive definite on TM_0.

The pair $F^n = (M, F)$ is called a Finsler space of dimension n. F is called fundamental function and g_{ij} is called the fundamental tensor of the Finsler space F^n.
The normalized supporting element l_i and angular metric tensor h_{ij} of F are defined, respectively as:

$$l_i = \frac{\partial F}{\partial y^i}, \quad h_{ij} = F\frac{\partial^2 F}{\partial y^i \partial y^j}. \tag{2.1}$$

Let F be a Finsler metric defined by $F = \sqrt[m]{A}$, where A is given

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by $A = a_{i_1 i_2 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}$ with $a_{i_1 \dots i_m}$ symmetric in all its indices [12]. Then F is called an m-th root Finsler metric. Clearly, A is homogeneous of degree m in y.
Let

$$A_i = \frac{\partial A}{\partial y^i},\ A_{ij} = \frac{\partial^2 A}{\partial y^i \partial y^j},\ A_{x^i} = \frac{\partial A}{\partial x^i},$$
$$A_0 = A_{x^i} y^i,\ \sigma_{x^i} = \frac{\partial \sigma}{\partial x^i}. \qquad (2.2)$$

Then the followings hold

$$g_{ij} = \frac{A^{\frac{2}{m}-2}}{m^2}\left[mAA_{ij} + (2-m)A_iA_j\right],$$

$$y^iA_i = mA,\ y^iA_{ij} = (m-1)A_j,$$
$$y_i = \frac{1}{m}A^{\frac{2}{m}-1}A_i,\ A^{ij}A_{jk} = \delta^i_k,$$
$$A^{ij}A_i = \frac{y^j}{m-1},\ A_iA_jA^{ij} = \frac{mA}{m-1}.$$

3 Fundamental metric tensors and Spray coefficients of generalized Kropina conformal change of m-th root metrics

The differentiation of equation (1.1) with respect to y^i, yields the normalized supporting element $\bar{l}_i$ which is given by

$$\bar{l}_i = \bar{F}\left[\frac{(k+1)}{mA}A_i - \frac{k}{\beta}b_i\right] \qquad (3.1)$$

and the angular metric tensor $\bar{h}_{ij}$ is given by

$$\bar{h}_{ij} = \bar{F}^2\left[\frac{(k+1)}{mA}A_{ij} + \frac{(k+1)(k+1-m)}{m^2A^2}A_iA_j - \frac{k(k+1)}{m\beta A}(A_ib_j + A_jb_i) + \frac{k(k+1)}{\beta^2}b_ib_j\right]. \qquad (3.2)$$

The fundamental metric tensor $\bar{g}_{ij}$ of Finsler space $\bar{F}^n$ can be given by

$\bar{g}_{ij} = \bar{h}_{ij} + \bar{l}_i\bar{l}_j$.

By using equations (3.1) and (3.2), we obtain metric tensor $\bar{g}_{ij}$ as

$$\bar{g}_{ij} = \bar{F}^2\left[\frac{(k+1)}{mA}A_{ij} + \frac{(k+1)(2k+2-m)}{m^2A^2}A_iA_j - \frac{2k(k+1)}{m\beta A}(A_ib_j + A_jb_i) + \frac{k(2k+1)}{\beta^2}b_ib_j\right]. \qquad (3.3)$$

Above equation can be rewritten as

$$\bar{g}_{ij} = \bar{F}^2\left[\frac{(k+1)}{mA}A_{ij} + \frac{(k+1)\{2k+2-(2k+1)m\}}{(2k+1)m^2A^2}A_iA_j + \left(\frac{2(k+1)\sqrt{k}}{\sqrt{2k+1}mA}A_i - \frac{\sqrt{k(2k+1)}}{\beta}b_i\right) \times \left(\frac{2(k+1)\sqrt{k}}{\sqrt{2k+1}mA}A_j - \frac{\sqrt{k(2k+1)}}{\beta}b_j\right)\right]. \qquad (3.4)$$

Let

$$H_{ij} = \frac{(k+1)}{mA}A_{ij} + \frac{(k+1)\{2k+2-(2k+1)m\}}{(2k+1)m^2A^2}A_iA_j. \qquad (3.5)$$

From [11],

$$A_{ij} = B_{ij} + \epsilon C_iC_j,$$

then

$$A^{ij} = B^{ij} - \frac{\epsilon C^iC^j}{1+\epsilon C^2},$$

where $C^2 = B^{ij}C_iC_j$ and $C^i = B^{ij}C_j$. By using above results, we obtain

$$H^{ij} = \frac{mA}{k+1}A^{ij} - \frac{\{2k+2-(2k+1)m\}}{(k+1)(m-1)}y^iy^j. \qquad (3.6)$$

Thus in view of equations (3.4) and (3.5), $\bar{g}_{ij}$ can be written as

$$\bar{g}_{ij} = \bar{F}^2\left[H_{ij} + K_iK_j\right],$$

where

$$K_i = \frac{2(k+1)\sqrt{k}}{\sqrt{2k+1}mA}A_i - \frac{\sqrt{k(2k+1)}}{\beta}b_i,$$

$$K_j = \frac{2(k+1)\sqrt{k}}{\sqrt{2k+1}mA}A_j - \frac{\sqrt{k(2k+1)}}{\beta}b_j.$$

By direct computation, we have

$$\bar{g}^{ij} = \frac{1}{\bar{F}^2}\left[w_0A^{ij} + w_1y^iy^j + w_2B^iB^j + (y^iB^j + y^jB^i)\right],$$

where,

$$w_0 = \frac{mA}{(k+1)}, \quad (3.7)$$

$$w_1 = -\frac{(2k+2)-(2k+1)m}{(k+1)(m-1)} - \frac{k(2k+1)m^2\beta^2}{X},$$

$$w_2 = -\left[\frac{k(2k+1)m^2(m-1)^2A^2}{X}\right],$$

$$w_3 = -\left[\frac{k(2k+1)m^2(m-1)^2A\beta}{X}\right],$$

$$X = (k+1)(m-1)\left[\beta^2\left\{k(2m-3) + (m-1-2k^2)\right\} + km(m-1)(2k+1)AB^2\right],$$

$$B^i = A^{ij}b_j, \quad B^2 = B^ib_i.$$

Thus, we have

Proposition 3.1 : The covariant metric tensor $\bar{g}_{ij}$ and the contravariant metric tensor $\bar{g}^{ij}$of generalized Kropina confomal trasformed Finsler space $\bar{F}^n$ are given as:

$$\bar{g}_{ij} = \bar{F}^2\left[\frac{(k+1)}{mA}A_{ij} + \frac{(k+1)(2k+2-m)}{m^2A^2}A_iA_j - \frac{2k(k+1)}{m\beta A}(A_ib_j + A_jb_i) + \frac{k(2k+1)}{\beta^2}b_ib_j\right].$$

and

$$\bar{g}^{ij} = \frac{1}{\bar{F}^2}\left[w_0A^{ij} + w_1y^iy^j + w_2B^iB^j + w_3(y^iB^j + y^jB^i)\right],$$

where $w_0, w_1, w_2, w_3, X, B^i$ and B^2 are given by equation (3.7).

In local coordinates, the geodesics of a Finsler metric $F = F(x,y)$ are characterized by

$$\frac{d^2x^i}{dt^2} + 2G^i(x, \frac{dx^i}{dt}) = 0, \quad (3.8)$$

where

$$G^i = \frac{1}{4}g^{il}\{[F^2]_{x^qy^l}y^q - [F^2]_{x^l}\} \quad (3.9)$$

are called spray coefficients.

For calculate the spray coefficients $\bar{G}^i$, we find

$$\left[\bar{F}^2\right]_{x^q} = \bar{F}^2\left[2\sigma_{x^q} + \frac{2(k+1)A_{x^q}}{mA} - \frac{2k\beta_{x^q}}{\beta}\right] \quad (3.10)$$

and

$$\left[\bar{F}^2\right]_{x^qy^l}y^q = \bar{F}^2\left[2\sigma_{0l} + \frac{2(k+1)A_{0l}}{mA} + \frac{(k+1)\left\{4(k+1)-2m\right\}}{m^2A^2}A_lA_0 - \frac{2k}{\beta}\beta_{0l} + \frac{2k(2k+1)}{\beta^2}b_l\beta_0 + \frac{4\sigma_0(k+1)A_l}{mA} - \frac{4k\sigma_0b_l}{\beta} - \frac{4k(k+1)}{m\beta A}(A_l\beta_0 + A_0b_l)\right]. \quad (3.11)$$

In view of equations (3.9), (3.10), (3.11) and proposition 3.1, we have

$$\begin{aligned}\bar{G}^i = \frac{1}{4}\big[&w_0 A^{il} + w_1 y^i y^l + w_2 B^i B^l \\ &+ (y^i B^l + y^l B^i)\big] \times \qquad (3.12) \\ &\Big[2\sigma_{0l} + \frac{2(k+1)A_{0l}}{mA} \\ &+\frac{(k+1)\{4(k+1)-2m\}}{m^2A^2}A_l A_0 \\ &-\frac{2k}{\beta}\beta_{0l} + \frac{2k(2k+1)}{\beta^2}b_l\beta_0 \\ &+\frac{4\sigma_0(k+1)A_l}{mA} - \frac{4k\sigma_0 b_l}{\beta} \\ &-\frac{4k(k+1)}{m\beta A}(A_l\beta_0 + A_0 b_l) \\ &-2\sigma_{x^l} - \frac{2(k+1)A_{x^l}}{mA} + \frac{2k\beta_{x^l}}{\beta}\Big].\end{aligned}$$

Proposition 3.2 : Let $\bar{F} = e^{\sigma(x)}\frac{F^{k+1}}{\beta^k}$ be a non-Riemannian generalized Kropina conformal change of m-th root metric F on a manifold of dimension $n \geq 2$, with $m \geq 3$. Then the spray coefficients $\bar{G}^i$ of $\bar{F}^n$ are given by equation (3.12).

Remark 3.1 We see that the metric tensors $\bar{g_{ij}}$ and $\bar{g^{ij}}$ of $\bar{F}^n$ are not necessarily rational functions of y, but spray coefficients $\bar{G}^i$ of $\bar{F}^n$ are rational functions of y.

4 Einstein metrics

In Finsler geometry, the flag curvature is an analogue of sectional curvature in Riemannian geometry. A natural problem is to study and characterize Finsler metrics of constant flag curvature. There are only three local Riemannian metrics of constant sectional curvature, up to a scaling. However there are lots of non-Riemannian Finsler metrics of constant flag curvature. For example, the Funk metric is positively complete and non-reversible with $K = \frac{1}{4}$ and the Hilbert-Klein metric is complete and reversible with $K = 1$. Clearly, if a Finsler metric is of constant flag curvature, then it is an Einstein metric.

For a Finsler metric F, the Riemann curvature $R_y : T_xM \to T_xM$ is defined by $R_y(u) = R^i_k(x,y)u^k\frac{\partial}{\partial x^i}$, $u = u^k\frac{\partial}{\partial x^i}$, where

$$\begin{aligned}R^i_k = 2\frac{\partial G^i}{\partial x^k} - y^j\frac{\partial^2 G^i}{\partial x^j\partial y^k} \\ +2G^j\frac{\partial^2 G^i}{\partial y^j\partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \quad (4.1)\end{aligned}$$

The Finsler metric $F = F(x,y)$ is said to be of scalar curvature if there is a scalar function $K = K(x,y)$ such that

$$R^i_k = K(x,y)F^2\left\{\delta^i_k - \frac{F_{y^k}y^i}{F}\right\}. \quad (4.2)$$

The Ricci curvature of Finsler metric F is a scalar function $Ric : TM \to R$, defined to be the trace of R_y, i.e.,

$$Ric(y) := R^k_k(x,y).$$

A Finsler metric F on an n-dimensional manifold M is called an Einstein metric if there is a scalar function $K = K(x)$ on M such that

$$Ric = K(n-1)F^2.$$

A Finsler metric is said to be Ricci-flat if $Ric = 0$.
In view of definition of Riemann curvature, Ricci curvature and remark 3.1, we have

Lemma 4.1 *Let $\bar{F} = e^{\sigma(x)}\frac{F^{k+1}}{\beta^k}$ be a non-Riemannian generalized Kropina conformal change of m-th root metric F on a manifold of dimension $n \geq 2$, with $m \geq 3$. Then $\bar{R^i_k}$ and $\bar{Ric} = R^k_k$ are rational function of y.*

By definition, every 2-dimensional Riemann metric is an Einstein metric, but generally not of Ricci constant. In dimension $n \geq 3$, the second Schur's Lemma ensures that every Riemannian Einstein metric must be Ricci constant. In particular, in dimension $n = 3$, a Riemann metric is Einstein if and only if it is of constant sectional curvature.

Proof of Theorem 1.1 By Lemma 4.1, $\bar{Ric}$ is a rational function of y. Suppose $\bar{F}$ is an Einstein metric, that is, $\bar{Ric} = K(n-1)\bar{F}^2$ and $\bar{F}^2$ is not a rational function because $m \nmid (k+1)$. Therefore $K = 0$. □
We obtain

Corollary 4.1 *Let* $\bar{F} = e^{\sigma(x)} \frac{F^{k+1}}{\beta^k}$ *be a non-Riemannian generalized Kropina conformal change of m-th root metric on a manifold of dimension* $n \geq 2$*, where* $m \geq 3$ *and* $m \nmid (k+1)$*. If* $\bar{F}$ *is of constant flag curvature* K*, then* $K = 0$.

5 Weak Einstein metrics

A weak Einstein metric is generalization of Einstein metric. A Finsler metric F is called a weak Einstein metric if its Ricci curvature Ric is in the form $Ric = (n-1)(\frac{3\theta}{F}+\lambda)F^2$, where θ is a 1-form and $\lambda = \lambda(x)$ is a scalar function. In general, a weak Einstein metric is not necessarily an Einstein metric and vice versa.
Proof of Theorem 1.2. Suppose $\bar{F}$ is a weak Einstein metric, then

$$Ric = (n-1)(3\theta\bar{F} + \lambda\bar{F}^2).$$

By Lemma 4.1, $\bar{Ric}$ is rational function of y. Therefore, we have:
If $\lambda \neq 0$, we get

$$\bar{F} = \frac{-3(n-1)\theta \pm \sqrt{9(n-1)^2\theta^2 + 4(n-1)\lambda\bar{Ric}}}{2(n-1)\lambda}.$$

On the other hand,
$\bar{F} = \frac{\left(a_{i_1 i_2 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}\right)^{\frac{k+1}{m}}}{\beta^k}$, so we get

$$\left(a_{i_1 i_2 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}\right)^{\frac{k+1}{m}} = \left(\frac{-3(n-1)\theta \pm \sqrt{9(n-1)^2\theta^2 + 4(n-1)\lambda\bar{Ric}}}{2(n-1)\lambda}\right)\beta^k.$$

Here the left hand side is purely irrational for $m \geq 3$ and $m \nmid (k+1)$. Then right hand side will be irrational if and only if $\theta = 0$. Thus we have, $\bar{F}$ is an Einstein metric. Using theorem 1.1, we obtain $\bar{Ric} = 0$. □

6 Scalar flag curvature

For a tangent plane P=span(y,u), where y and u are linearly independent vectors of tangent space T_xM of M at point $x \in M$, the flag curvature $\mathbf{K}(x, y, P)$ with pole vector y is defined by

$$\mathbf{K}(x, y, P) := \frac{g_y(R_y(u), u)}{g_y(y, y)g_y(u, u) - g_y(y, u)g_y(y, u)},$$

where $u \in P$.
If $\mathbf{K}(x, y, P) = \mathbf{K}(x, y)$, then the Finsler metric is said to be of scalar flag curvature.
If $\mathbf{K}(x, y, P) = \mathbf{K}(x)$, then the Finsler metric is said to be of isotropic flag curvature.
If $\mathbf{K}(x, y, P) = \frac{3\theta}{F} + c(x)$, where $c = c(x)$ is a scalar functions on M and θ is an exact form on M, then the Finsler metric F is said to of almost isotropic flag curvature.
That is, F is called of almost isotropic flag curvature if

$$K = \frac{3c_{x^m} y^m}{F} + \lambda, \qquad (6.1)$$

where $c = c(x)$ and $\lambda = \lambda(x)$ are some scalar functions on M.
If $\mathbf{K}(x, y, P) =$ constant, then the Finsler metric is said to be of constant flag curvature.
F is of weakly isotropic flag curvature if

$$K = \frac{3\theta}{F} + \lambda, \qquad (6.2)$$

where θ is an 1-form and $\lambda = \lambda(x)$ is a scalar function.
Clearly, if a Finsler metric is of weakly isotropic flag curvature, then it is a weak Einstein metric.

Lemma 6.1 *Let $\bar{F} = e^{\sigma(x)} \frac{F^{k+1}}{\beta^k}$ be a non-Riemannian generalized Kropina confomal change of m-th root metric on a manifold of dimension $n \geq 2$, where $m \geq 3$ and $m \nmid (k+1)$. If $\bar{F}$ is of almost isotropic flag curvature K, then $K = 0$.*

The S-curvature $S = S(x,y)$ in Finsler geometry has been introduced by Z. Shen [11] as a non-Riemannian quantity, defined as

$$S(x,y) = \frac{d}{dt}[\tau(\sigma(t), \dot{\sigma}(t))]_{|t=0}, \quad (6.3)$$

where $\tau = \tau(x,y)$ is a scalar function on $T_xM\backslash\{0\}$, called distortion of F and $\sigma = \sigma(t)$ be the geodesic with $\sigma(0) = x$ and $\dot{\sigma}(0) = y$.
A Finsler metric F is called of isotropic S-curvature if

$$S = (n+1)cF, \quad (6.4)$$

for some scalar function $c = c(x)$, on M.

Theorem 6.1 *([5]) Let (M, F) be an n-dimensional Finsler manifold of scalar flag curvature $K(x,y)$. Suppose that the S-curvature is isotropic, then there is a scalar function $\lambda(x)$ on M such that $K = \frac{3c_{x^m}y^m}{F} + \lambda$. In Particular, $c(x) = c$ is a constant if and only if $K = K(x)$ is a scalar function on M.*

In dimension $n \geq 3$, a Finsler metric F is of isotropic flag curvature if and only if F is of constant flag curvature by Schur's Lemma. In general, a Finsler metric of weakly isotropic flag curvature and that of isotropic flag curvature are not equivalent.
Proof of Theorem 1.3 By lemma 6.1 and theorem 6.1, complete the proof of the theorem 1.3. □

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