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COMMON FIXED POINT THEOREM FOR COINCIDENTALLY COMMUTING MAPPINGS IN MULTIPLICATIVE METRIC SPACES

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Abstract : In this paper ,we prove a unique common fixed point theorems using Presic type contraction in complete multiplicative metric spaces.

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Introduction and preliminaries.
The set of positive real numbers is not complete with respect to usual metric. To overcome this difficulty, in 2008, Bashirov et al. [5] introduced the concept of multiplicative metric spaces as follows:

Definition1.1. ([5]) Let X be a non-empty set. A multiplicative metric is a mapping $d: X\times X \to \mathbb{R}^+$ satisfying the following conditions:

(i) d(x, y) ≥ 1 for all x, y ∈ X and d(x, y) = 1 if and only if x=y;
(ii) d(x, y) = d(y, x) for all x, y ∈ X;
(iii) d(x, y) ≤ d(x, z). d(z, y) for all x, y, z ∈ X (multiplicative triangle inequality).

Then mapping d together with X i.e., (X, d) is known as multiplicative metric spaces.

Example1.2.([5]) Let R^n_+ be the collection of all n-tuples of positive real numbers.
Let d^*: $\mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}$ be defined as follows:

$$d^*(x, y) = \left(\left|\frac{x_1}{y_1}\right|^* . \left|\frac{x_2}{y_2}\right|^* \dots \left|\frac{x_n}{y_n}\right|^*\right),$$

where $x=(x_1, . . . ,x_n)$, $y=(y_1, . . . ,y_n) \in \mathbb{R}^n_+$ and $|.| : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$|a|^* = \begin{cases} a & if\ a \geq 1; \\ \frac{1}{a} & if\ a < 1. \end{cases}$$

Then (X, d) is a multiplicative metric space.

Example1.3. ([10]) Let d: $\mathbb{R} \times \mathbb{R} \to [1, \infty)$ be defined by

$$d(x, y) = a^{|x-y|}\ ,\text{where } x, y \in \mathbb{R} \text{ and } a > 1.$$

Then d(x, y) is multiplicative metric and (X, d) is a multiplicative metric space. We may call it usual multiplicative metric spaces.

In 2015, M. Abbas et.al. introduced the notion of multiplicative absolute value function as follow:

Definition 1.4.([2]) A multiplicative absolute value function $|\cdot|: \mathbb{R} \to \mathbb{R}^+$ is defined as

$$|x| = \begin{cases} x & if \quad x \geq 1 \\ \frac{1}{x} & if \quad x \in (0,1) \\ 1 & if \quad x = 0 \ . \\ -\frac{1}{x} & if \quad x \in (-1,0) \\ -x & if \quad x \leq -1 \end{cases}$$

Proposition 1.5.([2]) For arbitrary x, $y \in \mathbb{R}^+$,the multiplicative absolute value function $|\cdot| : \mathbb{R}^+ \to \mathbb{R}^+$satisfies the following:

(1) $|x| \geq 1$.
(2) $x \leq |x|$.
(3) $1/|x| \leq x$ if $x > 0$ and $x \leq 1/|x|$ if $x \leq 0$.
(4) $|x{\cdot}y| \leq |x||y|$.

One can refer to ([10]) for detailed multiplicative metric topology.

Definition1.6.([7]) Let (X, d) be a multiplicative metric space. A sequence $\{x_n\}$ in X said to be a

(i) multiplicative convergent sequence to x, if for every multiplicative open ball $B_\epsilon(x) = \{ y \mid d(x, y) < \epsilon\}$, $\epsilon > 1$, there exists a natural number N such that $x_n \in B_\epsilon(x)$ for all n ≥ N, i. e, $d(x_n, x) \to 1$ as $n \to \infty$.
(ii) multiplicative Cauchy sequence if for all $\epsilon > 1$, there exists N ∈ ℕ such that $d(x_n, x_m) < \epsilon$ for all m, n > N i. e , $d(x_n, x_m) \to 1$ as $n \to \infty$.

A multiplicative metric space is called complete if every multiplicative Cauchy sequence in X is multiplicative convergent to x ∈ X.

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In 2012, Ozavsar gave the concept of multiplicative contraction mapping and proved some fixed point theorem for these maps in complete multiplicative metric spaces.

Definition1.7.([7]) Let (X, d) be a multiplicative metric space. The map f : X → X is called a multiplicative contraction if there exists a real constant λ ∈ [0, 1) such that

(1.1) $d(f(x_1), f(x_2)) \leq (d(x_1, x_2))^{\lambda}$ for all x, y ∈ X.

Consider the k-th order nonlinear difference equation

x_{n+k}= f (x_n, …, x_{n+k-1}), n ∈ℕ with the initial values x_0, x_1, …, x_k ∈ X, where (X, d) is a metric space, k ∈ N, k ≥ 1 and f : X^k → X.

Equation (1.1) can be studied for fixed point theory in view of the fact that x^*∈ X is a solution of (1.1) if and only if x^* is a fixed point of f , that is, x^*= f (x^*, …, x^*).

Definition 1.8. Let (X, d) be a metric space, k a positive integer, and f: X^k→X and g:X → X mappings.

(b) An element $x \in X$ is said to be a fixed point of f if x = (x,…,).

(c) If x = gx = f(x,…,x),then x is called a common fixed point of $f$and$g$.

(d) Mappings $f$and $g$are said to be commuting if ((x,…,)) = f(gx,…,gx) , for all $x \in X$.

(f) Mappings f and g are said to be weakly commuting if

d(f(g(x, x, …x)),g(fx, fx, …fx)) ≤ d(f(x, x, …x),g(x, x, …x)) for all x ∈ X.

(b) An element $x \in X$ is said to be a coincidence point of $f$and g if gx = (x,…,).

(e) Mappings$f$ and $g$are said to be k-compatible (coincidentally commuting) if g (f(p, p, . . . , p)) = f(gp, gp, . . . , gp), whenever p ∈ X is such that gp = f (p, p, . . . , p).

Remark 1.09. The above definition are used in similar mode multiplicative metric spaces.

Remark 1.10. For k=1, the above definitions reduce to the usual definition of commuting and weakly compatible mappings in a multiplicative metric space.

In 1965, S.B. Presic in [8] gives the most important results on this direction by generalizing the Banach contraction mapping principle as follows:

Theorem 1.11. ([8]). Let (X, d) be a complete metric space, k a positive integer and T : X^k→ X a mapping satisfying the following contractive type condition

(1.2.1) d(T (x_1, x_2, …, x_k), T (x_2, x_3, …, x_{k+1})) ≤ q_1d(x_1, x_2) + q_2d(x_2, x_3) + … + q_kd(x_k, x_{k+1}) for every x_1, x_2, x_3, …, x_k, x_{k+1}in X, where q_1, q_2, …, q_k are non -negative constants such that q_1+ q_2+ …+ q_k < 1.

Then there exists a unique point x in X such that T (x, x, …, x) = x.

Moreover, if x_1, x_2, …, x_k are arbitrary points in X and for n ∈ N , x_{n+k} = T (x_n, x_{n+1}, …, x_{n+k-1}) then the sequence $\{x_n\}$ is convergent and limx_n =T (limx_n, limx_n, …, limx_n).

2. Main results

Now we prove above theorem in setting of multiplicative metric space as follows:

In 2011, R. George, M. S. Khan[6] proved following theorem in metric spaces as follows:

Theorem 2.1.[6] Let(X, d) be a metric space, k a positive integer, T: X^k→X and f: X→X be mappings satisfying the following conditions :

(2.1) T(X^k)⊂ f(x)

(2.2) d(T(x_1, x_2, x_3, . . . , x_k),T(x_2, x_3, x_4, . . . , x_k, x_{k+1})) ≤ λ max{d(fx_i, f x_{i+1}) : 1 ≤ i ≤ k}, where x_1, x_2, x_3, . . . , x_k, $x_{k+1}$are arbitrary elements in X and λ ∈ (0,1)

(2.3) f(X)is complete .

Then C(f, T) ≠ ∅.

Further, if f is idempotent at some u ∈ C(f, T), and f and T are coincidentally commuting then f and T has a common fixed point.

The set of coincidence points of f and T is denoted by C(f, T).

Theorem 2.2 Let (X, d) be a multiplicative metric space, k a positive integer, let T be a mapping of X^kinto X and let f be a mapping of X into X satisfying

(2.4) T (X^k) ⊆ f (X),

(2.5) f (X) is complete ,

(2.6) (f, T) is a are coincidentally commuting and f is idempotent ,

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(2.7) d(T (x_1, x_2, x_3, . . . , x_k), T (x_2, x_3, x_4, . . . , x_k, x_{k+1})) $\le [\max\{d(fx_i, f x_{i+1}) : 1 \le i \le k\}]^{\lambda}$, for all x_1, x_2, x_3, . . . , x_k, x_{k+1} ∈ X, where 0 < λ < 1,

(2. 8) d(T (u, u, . . . , u), T (v, v, . . . , v)) < d(f u, f v), for all distinct u, v ∈ X.

Then there exists a unique point z ∈ X such that fz = z = T (z, z, . . . , z).

Proof. Let x_1, x_2, x_3, . . . , x_k be arbitrary elements in X. By (2.4), we define a sequence $\{y_n\}$in f(X) as follows :

y_{n+k}= f(x_{n+k})=T(x_n, x_{n+1}, … x_{n+k-1}), for n=1,2, ...

For simplicity set α_n = d(y_n, y_{n+1}). We shall prove by induction that for each n ∈ N:

$\alpha_n \le K^{\theta^n}$ (where $\theta = (\lambda)^{1/k} < 1$, $K = \max\{(\alpha_1)^{1/\theta}, (\alpha_2)^{1/\theta^2}, \ldots, (\alpha_k)^{1/\theta^k}\}$

According to the definition of K we see that (2.11) is true for n = 1, . . . , k.

Now let the following k inequalities: $\alpha_n \le K^{\theta^n}$, $\alpha_{n+1} \le K^{\theta^{n+1}}$, . . . , $\alpha_{n+k-1} \le K^{\theta^{n+k-1}}$ be the induction hypotheses.

Then we have:

α_{n+k} = d(y_{n+k}, y_{n+k+1}) = d(T (x_n, x_{n+1}, . . . , x_{n+k-1}), T (x_{n+1}, x_{n+2}, . . . , x_{n+k}))

$\le \max\{d(fx_n, fx_{n+1}), d(f x_{n+1}, f x_{n+2}), \ldots d(f x_{n+k-1}, f x_{n+k})\}^{\lambda}$

$= [\max\{\alpha_n, \alpha_{n+1}, \ldots \alpha_{n+k-1}\}]^{\lambda}$

$\le [\max\{K^{\theta^n}, K^{\theta^{n+1}}, \ldots K^{\theta^{n+k-1}}\}]^{\lambda}$

$\le [K^{\theta^n}]^{\lambda}$ {as θ < 1}

$= K^{\theta^{n+k}}$ {as $\lambda = \theta^k$}

Thus inductive proof of (2.5) is complete. Now, for n, p ∈ N, we have

d(y_n, y_{n+p}) ≤ d(y_n, y_{n+1}).d(y_{n+1}, y_{n+2}).....d(y_{n+p-1}, y_{n+p})

$\le K^{\theta^n} . K^{\theta^{n+1}} ..... K^{\theta^{n+p-1}}$

$\le K^{\theta^n(1+\theta+\theta^2+\cdots)}$

$\le K^{\theta^n/1-\theta}$.

Hence sequence $\{y_n\}$ is a Cauchy sequence in f(X). As f(X) is complete, there exists z ∈ f(X) such that $\lim_{n\to\infty} y_n$= z. Hence there exists a point p ∈ X such that z = f p.

Now consider

d(f x_{n+k}, T (p, p, . . . , p)) = d(T (p, p, . . . , p), T (x_n, x_{n+1}, . . . , x_{n+k-1}))

≤ d(T (p, p, . . . , p), T (p, p, . . . , p, x_n)).d(T (p, p, . . . , p, x_n), T (p, p, . . . , p, x_n, x_{n+1}))

. d(T (p, p, . . . , p, x_n, x_{n+1}), T (p, p, . . . , p, x_n, x_{n+1}, x_{n+2})).d(T (p, p, . . . , p, x_n, x_{n+1}, x_{n+2}), T (p, p, . . . , p, x_n, x_{n+1}, x_{n+2}, x_{n+3})). . . . . d(T (p, x_n, x_{n+1}, . . . , x_{n+k-2}), T (x_n, x_{n+1}, . . . , x_{n+k-1})).

$\le [d(f p, fx_n)]^{\lambda}. [\max\{d(fp, fx_n), d(fx_n, f x_{n+1})\}]^{\lambda}$
$.[\max\{d(f p, f x_n), d(f x_n, f x_{n+1}), d(f x_{n+1}, f x_{n+2})\}]^{\lambda}.$
$[\max\{d(f p, fx_n), d(fx_n, f x_{n+1}), d(f x_{n+1}, f x_{n+2}), d(f x_{n+2}, f x_{n+3})\}]^{\lambda}$
. . . .
.
$[\max\{d(f p, f x_n), d(fx_n, f x_{n+1}), \ldots, d(f x_{n+k-2}, f x_{n+k-1})\}]^{\lambda}$
.

Letting n → ∞, we get

d(fp, T (p, p, . . . , p)) ≤ 1, so that f p = T (p, p, . . . , p).

Now suppose that f is idempotent, while f and T are coincidentally commuting pair. Then ,we have ffp = fp and f (T (p, p, . . . , p)) = T(f p, f p, . . . , f p). Therefore, fp = ffp= f (T (p, p, . . . , p)) = T(f p, f p, . . . , f p). Thus fp is a common fixed point of f and T .We now have z = fz = T (z, z, . . . , z).

Uniqueness:

Suppose that there exists a point $z' \ne z$ in X such that z'= fz'= T (z', z', . . . , z').

Then d(z, z') = d(T (z, z, . . . , z), (T (z', z', . . . , z')) < d(f z, fz') from (2.8) = d(z, z'),which is a contradiction.

Therefore z = z'. Hence z is the unique point fixed point.

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