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SOME FIXED POINT THEOREMS IN MULTIPLICATIVE METRIC SPACES SATISFYING RATIONAL INEQUALITIES

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Abstract: In this paper, we prove some fixed point theorems using rational inequality in multiplicative metric spaces.

Key words and Phrases: Multiplicative metric spaces, rational inequality, fixed point.

Mathematics Subject Classification: 47H10, 54H25.

1. Introduction and preliminaries

It is well know that the set of positive real numbers $\mathbb{R}_+$ is not complete according to the usual metric. To overcome this problem, in 2008, Bashirov et al. [2] introduced the concept of multiplicative metric spaces as follows:

Definition1.1. ([2]) Let X be a non-empty set. A multiplicative metric is a mapping d: X×X → $\mathbb{R}^+$ satisfying the following conditions:

(i) d(x, y) ≥ 1 for all x, y ∈ X and d(x, y) = 1 if and only if x=y;

(ii) d(x, y) = d(y, x) for all x, y ∈ X;

(iii) d(x, y) ≤ d(x, z). d(z, y) for all x, y, z ∈ X (multiplicative triangle inequality).

Then mapping d together with X i.e., (X, d) is a multiplicative metric space.

Example1.2.([8]) Let R^n_+ be the collection of all n-tuples of positive real numbers.

Let d^*: $\mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}$ be defind as follows:

$$d^*(x, y) = \left(\left|\frac{x_1}{y_1}\right|^* . \left|\frac{x_2}{y_2}\right|^* \dots \left|\frac{x_n}{y_n}\right|^*\right),$$

where x=$(x_1, . . . ,x_n)$, y=$(y_1, . . . ,y_n)$ ∈ $\mathbb{R}^n_+$ and $|.|$: $\mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$|a|^* = \begin{cases} a & if\ a \geq 1; \\ \frac{1}{a} & if\ a < 1. \end{cases}$$

Then it is obvious that all conditions of multiplicative metric are satisfied.

Example1.3. ([10]) Let d: ℝ × ℝ→ [1, ∞) be defined as

d(x, y) = $a^{|x-y|}$,where x, y ∈ ℝ and a > 1.

Then d(x, y) is a multiplicative metric and (X, d) is called a multiplicative metric space. We call it usual multiplicative metric spaces.

Example1.4.([10]) Let (X, d) be a metric space .Define a mapping d_a on X by $d_a(x, y) = a^{d(x,y)}$ where a > 1 is a real number and $d_a(x, y) = a^{d(x,y)} = \begin{cases} 1 & if\ x = y \\ a & if\ x \neq y. \end{cases}$

The metric $d_a(x, y)$ is called discrete multiplicative metric and X together with metric d_a i.e., (X, d_a) is known as a discrete multiplicative metric space.

Example 1.5.([1]) Let $X = C^*[a, b]$ be the collection of all real-valued multiplicative continuous functions over$[a, b] \subseteq R^+$. Then (X, d) is a multiplicative metric space with metric d defined by

$$d(f,g) = \sup_{x\in[a,b]} \left|\frac{f(x)}{g(x)}\right| \text{ for } f,g \in X.$$

Remark1.6. We note that the example 1.1 is valid for positive real numbers and example 1.2 is valid for all real numbers.

Remark 1.7.([10]) Neither every metric is multiplicative metric nor every multiplicative metric is metric. The mapping d^* defined above is multiplicative metric but not metric as it doesn't satisfy triangular inequality. Consider $d^*(\frac{1}{3}, \frac{1}{2}) + d^*(\frac{1}{2}, 3) = \frac{3}{2} + 6 = 7.5 < 9 = d^*(\frac{1}{3}, 3)$.

On the other, hand the usual metric on R is not multiplicative metric as it doesn't satisfy multiplicative triangular inequality, since d(2, 3) · d(3, 6) = 3 < 4 = d(2, 6).

One can refer to ([8]) for detailed multiplicative metric topology.

Definition1.8.([8]) Let (X, d) be a multiplicative metric space. A sequence $\{x_n\}$ in X said to be a

(i) multiplicative convergent sequence to x, if for every multiplicative open ball $B_\epsilon(x) = \{ y \mid d(x, y) < \epsilon\}$, $\epsilon > 1$, there exists a natural number N such that $x_n \in B_\epsilon(x)$ for all n ≥ N, i. e, $d(x_n, x) \to 1$ as $n \to \infty$.

(ii) multiplicative Cauchy sequence if for all $\epsilon > 1$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all m, n > N i. e , $d(x_n, x_m) \to 1$ as $n \to \infty$.

A multiplicative metric space is called complete if every multiplicative Cauchy sequence in X is multiplicative converging to x ∈ X.

Remark1.9. We note that the set of positive real numbers $\mathbb{R}_+$ is not complete according to the usual metric. Let X = $\mathbb{R}_+$ .Consider the sequence $x_n = \{\frac{1}{n}\}$. It is obvious $\{x_n\}$ is a Cauchy sequence in X with respect to usual metric spaces X and it is not complete metric space as every Cauchy sequence in X does not converge in $\mathbb{R}_+$ i.e., $0 \notin \mathbb{R}_+$. In case of multiplicative metric spaces, consider the sequence $x_n = \{a^{1/n}\}$,where $a > 1$, it is complete in multiplicative metric spaces, since for n ≥ m,

$$d^*(x_n, x_m) = \left|\frac{x_n}{x_m}\right|^* = \left|\frac{a^{1/n}}{a^{1/m}}\right|^* = \left|a^{\frac{1}{n}-\frac{1}{m}}\right|^* = a^{\frac{1}{m}-\frac{1}{n}} < a^{\frac{1}{m}}$$

$< \epsilon$ if $m > \frac{\log a}{\log \epsilon}$,

where $|a|^* = \begin{cases} a & \text{if } a \geq 1; \\ \frac{1}{a} & \text{if } a < 1. \end{cases}$

This implies $\{x_n\}$ is a Cauchy sequence in X and it converges to $1 \in \mathbb{R}_+$ as $n \to \infty$. Hence (X, d) is a complete multiplicative metric space.

In 2012, Özavşar and Çevikel[8] introduced the concepts of Banach-contraction, Kannan-contraction, and Chatterjea-contraction mappings in the sense of multiplicative metric spaces as follows:

(Banach-contraction). Let (X, d) be a complete multiplicative metric space and let f: X → X be a multiplicative contraction if there exists a real constant λ ∈ [0, 1) such that

$$d(f(x), f(y)) \leq d(x, y)^\lambda \text{ for all } x, y \in X.$$

Then f has a unique fixed point.

(Kannan-contraction). Let (X, d) be a complete multiplicative metric space. Suppose the mapping f : X → X satisfies the contraction condition

$$d(fx, fy) \leq (d(fx, x) \cdot d(fy, y))^\lambda, \text{ for all } x, y \in X, \text{ where } \lambda \in [0, \tfrac{1}{2}).$$

Then f has a unique fixed point in X and for any x ∈ X, iterative sequence $(f_n(x))$ converges to the fixed point.

(Chatterjea-contraction). Let (X, d) be a complete multiplicative metric space. Suppose the mapping f : X → X satisfies the contraction condition d(fx,fy) ≤ (d(fy,x) · d(fx,y))$^{\lambda}$, for all x, y ∈ X, where λ ∈ [0, $\frac{1}{2}$).

Then f has a unique fixed point in X and for any x ∈ X, iterative sequence $(f_n(x))$ converges to the fixed point.

2. Main results.

Now we prove some fixed point theorems for a map that satisfy various types of rational inequalities.

Theorem 2.1. Let (X, d) be a complete multiplicative metric space. Suppose the mapping

f : X → X be a continuous self- mapping satisfies the condition

d(fx,fy)≤

$$[d(x,y)]^{a_1}.[d(x,fy)]^{a_2}.[d(fx,y)]^{a_3}.[d(fy,y)]^{a_4}.[\frac{d(y,fy)d(x,fx)}{d(x,y)}]^{a_5}.[\frac{d(y,fx)d(x,fx)}{d(x,y)d(y,fy)}]^{a_6},$$

for all x, y ∈ X, where $a_1,\ a_2, a_3, a_4, a_5, a_6 \geq 0$ and $a_1 + 2a_2 + 2a_3 + a_4 + a_5 + a_6 < 1$

Then f has a unique fixed point in X.

Proof. Let $\{x_n\}$be a sequence in X defined as follows.

Let x_0 ∈ X. For this x_0 there exists x_1 such that f(x_0) = x_1. Again, for this x_1 there exists x_2 such that f (x_1) = x_2. Continue like this we get f (x_n) = x_{n+1}.

Consider

d(x_n , x_{n+1}) = d(Tx_{n-1},Tx_n)

$$\leq [d(x_{n-1},x_n)]^{a_1}.[d(x_{n-1},fx_n)]^{a_2}.[d(fx_{n-1},x_n)]^{a_3}.[d(fx_n,x_n)]^{a_4}.[\frac{d(x_n,fx_n)d(x_{n-1},fx_{n-1})}{d(x_{n-1},x_n)}]^{a_5}.[\frac{d(x_n,fx_{n-1})d(x_{n-1},fx_{n-1})}{d(x_{n-1},x_n)d(x_n,fx_n)}]^{a_6}$$

$$\leq [d(x_{n-1},x_n)]^{a_1}.[d(x_{n-1},x_{n+1})]^{a_2}.[d(x_n,x_n)]^{a_3}.[d(x_{n+1},x_n)]^{a_4}.[d(x_{n+1},x_n)]^{a_5}.[d(x_{n-1},x_n)]^{a_6}$$

$$\leq 6.[d(x_n,x_{n+1})]^{a_3}.[d(x_{n+1},x_n)]^{a_4}.[d(x_{n+1},x_n)]^{a_5}.[d(x_{n-1},x_n)]^{a_6}$$

$\leq [d(x_{n-1},x_n)]^{a_1+a_2+a_3+a_6}.\ [d(x_{n+1},x_n)]^{a_2+a_3+a_4+a_5}$

d(x_n , x_{n+1}) ≤ $[d(x_{n-1},x_n)]^h$,

where h = $\frac{a_1+a_2+a_3+a_6}{1-(a_2+a_3+a_4+a_5)}$ < 1.

Similarly, d(x_{n-1}, x_n) ≤ $[d(x_{n-2},x_{n-1})]^h$,

d(x_n , x_{n+1}) ≤ $[d(x_{n-2},x_{n-1})]^{h^2}$.

Continue like this we get,

d(x_n , x_{n+1}) ≤ $[d(x_0,x_1)]^{h^n}$

For n > m, d(x_n , x_m) ≤ d(x_n , x_{n-1}) · d(x_{n-1}, x_{n-2}) · · · d(x_m , x_{m+1})

≤ d(x_0,x_1) $^{h^{n-1}+h^{n-2}+\cdots h^m}$

≤ d(x_0,x_1) $^{\frac{h^m}{1-h}}$. This implies d(x_n , x_m) →1 as n, m → ∞.

Hence (x_n) is a Cauchy sequence. By the multiplicative completeness of X, there is z ∈ X such that x_n → z as n →∞.

Now we show that z is fixed point of f by assuming f is continuous or not continuous.

(i) f is continuous, since x_n → z (n →∞) and f is continuous so, $\lim_{n\to\infty} f\,x_n$= fz = $\lim_{n\to\infty} x_{n+1}$= z, i.e., z is a fixed point of f.

(ii) f is not continuous then

d(fz, z) ≤ d(fx_n ,fz). d(fx_nz)

$$\leq [d(z,x_n)]^{a_1}.[d(x_n,fz)]^{a_2}.[d(fx_n,z)]^{a_3}.[d(fz,z)]^{a_4}.[\frac{d(z,fz)d(x_n,fx_n)}{d(z,x_n)}]^{a_5}.[\frac{d(z,fx_n)d(x_n,fx_n)}{d(z,x_n)d(z,fz)}]^{a_6}.$$

d(fz, z) ≤ $[d(z,fz)]^{a_2+a_4+a_5-a_6}$ gives fz = z, i.e., z is a fixed point of f.

Uniqueness: Suppose z, w (z ≠ w) be two fixed point of f, then

d(z, w) = d(fz, fw)

$$\leq [d(z,w)]^{a_1}.[d(z,fw)]^{a_2}.[d(fz,w)]^{a_3}.[d(fw,w)]^{a_4}.[\frac{d(w,fw)d(z,fz)}{d(z,w)}]^{a_5}.\left[\frac{d(w,fz)d(z,fz)}{d(z,w)d(w,fw)}\right].^{a_6}$$

d(z, w) ≤ $[d(z,w)]^{a_1+a_2+a_3}$ this implies that d(z, w) = 1 i.e., z = w.

Hence f has a unique fixed point .

Cor.1. Putting $a_2 = a_3 = a_4 = a_5 = a_6 = 0$ gives Banach-contraction[8] in the sense of multiplicative metric spaces.

Let (X, d) be a complete multiplicative metric space and let f: X → X be a multiplicative contraction if there exists a real constant a_1 ∈ [0, 1) such that

d(f(x), f(y)) ≤ $d(x,y)^{a_1}$ for all x, y ∈ X.

Then f has a unique fixed point.

Cor.2. Putting $a_2 = a_3 = a_4 = a_6 = 0,\ a_1 = a_5$ gives Kannan-contraction[8] in the sense of multiplicative metric spaces.

Let (X, d) be a complete multiplicative metric space. Suppose the mapping f : X → X satisfies the contraction condition

d(fx,fy) ≤ (d(fx,x) · d(fy,y))^{a_1}, for all x, y ∈ X, where a_1 ∈ [0, $\frac{1}{2}$).

Then f has a unique fixed point in X and for any x ∈ X, iterative sequence $(f_n(x))$ converges to the fixed point.

Cor.3. Putting $a_1 = a_4 = a_5 = a_6 = 0$, $a_2 = a_3$ gives Chatterjea-contraction[8] in the sense of multiplicative metric spaces.

Let (X, d) be a complete multiplicative metric space. Suppose the mapping f : X → X satisfies the contraction condition

d(fx,fy) ≤ $(d(fy,x) \cdot d(fx,y))^{a_2}$, for all x, y ∈ X, where $a_2 \in [0, \frac{1}{2})$.

Then f has a unique fixed point in X and for any x ∈ X, iterative sequence $(f_n(x))$ converges to the fixed point.

Cor.4. Putting $a_2 = a_3 = a_6 = 0$, gives Kholi results[7] in the sense of multiplicative metric spaces as follows:

Let (X, d) be a complete multiplicative metric space. Suppose the mapping f : X → X be a continuous self- mapping satisfies the condition

d(fx, fy)≤ $[d(x,y)]^{a_1}.[d(fy,y)]^{a_4}.[\frac{d(y,fy)d(x,fx)}{d(x,y)}]^{a_5}$, for all x, y ∈ X, where $a_1, a_4, a_5 \geq 0$ and $a_1 + a_4 + a_5 < 1$. Then f has a unique fixed point in X.

Cor.5. Putting $a_4 = a_5 = a_6 = 0$, gives Isufati results [5] in the sense of multiplicative metric spaces as follows:

Let (X, d) be a complete multiplicative metric space. Suppose the mapping f : X → X be a continuous self- mapping satisfies the condition

d(fx,fy)≤ $[d(x,y)]^{a_1}.[d(x,fy)]^{a_2}.[d(fx,y)]^{a_3}$,

for all x, y ∈ X, where $a_1, a_2, a_3 \geq 0$ and $a_1 + 2a_2 + 2a_3 < 1$.

Then f has a unique fixed point in X.

Cor.6. Putting $a_2 = a_3 = a_4 = a_6 = 0$, gives Jaggi results[6] in the sense of multiplicative metric spaces as follows:

Let (X, d) be a complete multiplicative metric space. Suppose the mapping f : X → X be a continuous self- mapping satisfies the condition

d(fx,fy)≤ $[d(x,y)]^{a_1}.[\frac{d(y,fy)d(x,fx)}{d(x,y)}]^{a_5}$, for all x, y ∈ X, where $a_1, a_5 \geq 0$ and $a_1 + a_5 < 1$

Then f has a unique fixed point in X.

Cor.7. Putting $a_4 = a_5 = a_6 = 0$, $a_2 = a_3$ gives Reich results [9] in the sense of multiplicative metric spaces as follows:

Let (X, d) be a complete multiplicative metric space. Suppose the mapping f : X → X be a continuous self- mapping satisfies the condition

d(fx,fy)≤ $[d(x,y)]^{a_1}.[d(x,fy).d(fx,y)]^{a_2}$, for all x, y ∈ X, where $a_1, a_2 \geq 0$ and $a_1 + 4a_2 < 1$. Then f has a unique fixed point in X.

Theorem 2.2. Let (X, d) be a complete multiplicative metric space. Suppose the mapping f : X → X be a continuous self- mapping satisfies the condition

d(fx, fy) ≤ $[d(x,y)]^{a_1}.[\frac{d(y,fx)d(x,fy)}{d(x,y)}]^{a_2}$,

for all x, y ∈ X, where $a_1, a_2 \geq 0$ and $a_1 + a_2 < 1$

Then f has a unique fixed point in X.

Proof. Let $\{x_n\}$ be a sequence in X, defined as follows:

Let $x_0 \in X$, f $(x_0) = x_1$, f$(x_1) = x_2, \cdots,$f$(x_n) = x_{n+1}$.

Consider

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$$
$$\leq [d(x_{n-1},x_n)]^{a_1}.[\frac{d(x_n,fx_{n-1})d(x_{n-1},fx_n)}{d(x_{n-1},x_n)}]^{a_2}$$
$$\leq [d(x_{n-1},x_n)]^{a_1}.[\frac{d(x_{n-1},x_{n+1})}{d(x_{n-1},x_n)}]^{a_2}$$
$$\leq [d(x_{n-1},x_n)]^{a_1-a_2}.[d(x_{n-1},x_n).d(x_{n+1},x_n)]^{a_2}$$
$$\leq [d(x_{n-1},x_n)]^{a_1+a_2+a_3+a_6}.[d(x_{n+1},x_n)]^{a_2+a_3+a_4+a_5}$$

$d(x_n, x_{n+1}) \leq [d(x_{n-1},x_n)]^h$,

where h = $\frac{a_1}{1-a_2} < 1$.

Similarly, d $(x_{n-1}, x_n) \leq [d(x_{n-2},x_{n-1})]^h$,

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$d(x_n, x_{n+1}) \leq [d(x_{n-2}, x_{n-1})]^{h^2}$.

Continue like this we get,

$d(x_n, x_{n+1}) \leq [d(x_0, x_1)]^{h^n}$

For n > m, $d(x_n, x_m) \leq d(x_n, x_{n-1}) \cdot d(x_{n-1}, x_{n-2}) \cdot \cdot \cdot d(x_m, x_{m+1})$

$\leq d(x_0, x_1)^{h^{n-1}+h^{n-2}+\cdots h^m}$

$\leq d(x_0, x_1)^{\frac{h^m}{1-h}}$. This implies $d(x_n, x_m) \to 1$ as n, m → ∞.

Hence (x_n) is a Cauchy sequence. By the multiplicative completeness of X, there is z ∈ X such that $x_n \to z$ (n →∞).

Now we show that z is fixed point of f.

Since f is continuous and $x_n \to z$ (n →∞) so, $\lim_{n\to\infty} f\, x_n$ = fz = $\lim_{n\to\infty} x_{n+1}$ = z,

i.e., z is a fixed point of f.

Uniqueness: Suppose z, w (z ≠ w) be two fixed point of f, then

d(z, w) = d(fz, fw)

$\leq [d(z,w)]^{a_1}.[\frac{d(w,z)d(z,w)}{d(z,w)}]^{a_2}$

$d(z, w) \leq [d(z,w)]^{a_1+a_2}$ this implies that d(z, w) = 1 i.e., z = w.

Hence f has a unique fixed point .

Cor.1. Putting a_2 = 0, gives Banach-contraction[8] results in the sense of multiplicative metric spaces as follows:

Let (X, d) be a complete multiplicative metric space and let f: X → X be a multiplicative contraction if there exists a real constant $a_1 \in$ [0, 1) such that

$d(f(x), f(y)) \leq d(x,y)^{a_1}$ for all x, y ∈ X.

Then f has a unique fixed point.

Cor.2. Putting $d(x,y) = 1$, gives Chatterjea-contraction[8] results in the sense of multiplicative metric spaces as follows:

Let (X, d) be a complete multiplicative metric space. Suppose the mapping f : X → X satisfies the contraction condition

$d(fx,fy) \leq (d(fy,x) \cdot d(fx,y))^{a_2}$, for all x, y ∈ X, where $a_2 \in [0, \frac{1}{2})$.

Then f has a unique fixed point in X and for any x ∈ X, iterative sequence $(f_n(x))$ converges to the fixed point.

Theorem 2.3. Let f be a continuous self-mapping defined on a complete multiplicative metric space X, further f satisfies the following conditions

$d(fx, fy) \leq [d(x, fx) . d(y, fy)]^{a_1}. [d(x, fy). d(y, fx)]^{a_2}. [d(x, y)]^{a_3}. [\frac{d(x,fx)\, d(y,Ty)}{d(x,y)}]^{a_4}.$

$\{ \max \{d(x, fx), d(y, fy), d(x, fy), d(y, fx), \frac{d(x,fx).d(y,fy).d(y,fx)}{d(x,y)}\}\}^{a_5}$

for all x, y ∈ X and $2a_1 + 2a_2 + a_3 + a_4 + a_5 < 1$ where $a_1, a_2, a_3, a_4, a_5 \in [0,1]$.

Then T has unique fixed point.

Proof. Let $\{x_n\}$ be a sequence in X, defined as follows:

Let $x_0 \in X$, $f(x_0) = x_1$, $f(x_1) = x_2$, ⋯, $f(x_n) = x_{n+1}$.

If $x_n = x_{n+1}$ for some n∈ N then x_n is a fixed point of f.

Taking $x_n \neq x_{n+1}$ for all n ∈ N

Consider

$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1})$

$\leq [d(x_n, fx_n) . d(x_{n-1}, fx_{n-1})]^{a_1}. [d(x_n, fx_{n-1}). d(x_{n-1}, fx_n)]^{a_2}.[d(x_n, x_{n-1})]^{a_3}. [\frac{d(x_n,fx_n)\, d(x_{n-1},Tx_{n-1})}{d(x_n,x_{n-1})}]^{a_4}.$

$\{ \max \{d(x_n, fx_n), d(x_{n-1}, fx_{n-1}), d(x_n, fx_{n-1}), d(x_{n-1}, fx_n), \frac{d(x_n,fx_n).d(x_{n-1},fx_{n-1}).d(x_{n-1},fx_n)}{d(x_n,x_{n-1})}\}\}^{a_5}$

$\leq [d(x_n, x_{n+1}) . d(x_{n-1}, x_n)]^{a_1}. [d(x_n, x_n). d(x_{n-1}, x_n)]^{a_2}.[d(x_n,$ $xn-1)]a3.[d(xn,xn+1)\, d(xn-1,xn)d(xn,xn-1)]a4.$

$\{ \max \{d(x_n, x_{n+1}), d(x_{n-1}, x_n), d(x_n, x_n), d(x_{n-1}, x_{n+1}), \frac{d(x_n,x_{n+1}).d(x_{n-1},x_n).d(x_{n-1},x_{n+1})}{d(x_n,x_{n-1})}\}\}^{a_5}$

$\leq [d(x_n, x_{n+1}) . d(x_{n-1}, x_n)]^{a_1}. [d(x_{n+1}, x_n). d(x_{n-1}, x_n)]^{a_2}.[d(x_n, x_{n-1})]^{a_3}. [d(x_n, x_{n+1})]^{a_4}.[d(x_n, x_{n+1})^2. d(x_{n-1}, x_n)]^{a_5}$

$d(x_{n+1}, x_n) \leq [d(x_n, x_{n+1})]^{a_1+a_4+a_2+2a_5}. [d(x_n, x_{n-1})]^{a_1+a_2+a_5+a_3}$,

$d(x_n, x_{n+1}) \leq [d(x_{n-1}, x_n)]^h$,

where h = $\frac{a_1+a_2+a_5+a_3}{1-(a_1+a_4+a_2+2a_5)} < 1$.

Similarly, $d(x_{n-1}, x_n) \leq [d(x_{n-2}, x_{n-1})]^h$,

$d(x_n, x_{n+1}) \leq [d(x_{n-2}, x_{n-1})]^{h^2}$.

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Continue like this we get,
$d(x_n, x_{n+1}) \leq [d(x_0, x_1)]^{h^n}$
For n > m, $d(x_n, x_m) \leq d(x_n, x_{n-1}) \cdot d(x_{n-1}, x_{n-2}) \cdot \cdot \cdot d(x_m, x_{m+1})$
$\leq d(x_0, x_1)^{h^{n-1}+h^{n-2}+\cdots h^m}$
$\leq d(x_0, x_1)^{\frac{h^m}{1-h}}$. This implies $d(x_n, x_m) \to 1$(n, m $\to \infty$).
Hence (x_n) is a Cauchy sequence. By the multiplicative completeness of X, there is z ∈ X such that $x_n \to z$ (n $\to\infty$).
Now we show that z is fixed point of f.
Since f is continuous and $x_n \to z$ (n $\to\infty$) so,
$\lim_{n\to\infty} f\, x_n$ = fz = $\lim_{n\to\infty} x_{n+1}$ = z,
i.e., z is a fixed point of f.

Uniqueness: Suppose z, w (z ≠ w) be two fixed point of f, then
d(v, w) = d(fv, fw)
$\leq [d(v,fv) . d(w,fw)]^{a_1} . [d(v,fw). d(w,fv)]^{a_2} . [d(v, w)]^{a_3} . [\frac{d(v,fv)\, d(w,Tw)}{d(v,w)}]^{a_4}$.

$\{ \max \{d(v,fv), d(w,fw), d(v,fw), d(w,fv), \frac{d(v,fv).d(w,fw).d(w,fv).}{d(v,w)}\}\}^{a_5}$
$d(v, w) \leq [d(v,w)]^{a_3+2a_2+a_5-a_4}$ this implies that d(v, w) = 1 i.e., v = w.
Hence f has a unique fixed point .

Cor.1. Putting $a_2 = a_3 = a_4 = a_5 = 0$ gives Kannan-contraction[8] in the sense of multiplicative metric spaces.
Let (X, d) be a complete multiplicative metric space. Suppose the mapping f : X → X satisfies the contraction condition

$d(fx,fy) \leq (d(fx,x) \cdot d(fy,y))^{a_1}$, for all x, y ∈ X, where $a_1 \in [0, \frac{1}{2})$.

Then f has a unique fixed point in X.

Cor.2. Putting $a_2 = a_4 = a_5 = 0$, gives Fisher-contraction [4] in the sense of multiplicative metric spaces as follows:
Let f be a continuous self- mapping defined on a complete multiplicative metric space X, further f satisfies the following conditions
$d(fx, fy) \leq [d(x,fx) . d(y,fy)]^{a_1} . [d(x, y)]^{a_3}$,for all x, y ∈ X and $2a_1 + a_3 < 1$, where $a_1, a_3 \in [0,1]$.
Then T has unique fixed point.

Cor.3. Putting $a_2 = a_3 = a_4 = a_5 = 0$, gives Chatterjea-contraction[8] in the sense of multiplicative metric spaces.
Let (X, d) be a complete multiplicative metric space. Suppose the mapping f : X → X satisfies the contraction condition

$d(fx,fy) \leq (d(fy,x) \cdot d(fx,y))^{a_1}$, for all x, y ∈ X, where $a_1 \in [0, \frac{1}{2})$.

Then f has a unique fixed point in X.

Cor.4. Putting $a_1 = a_2 = a_4 = a_5 = 0$, gives Banach-contraction[8] in the sense of multiplicative metric spaces as follows:
Let (X, d) be a complete multiplicative metric space and let f: X → X be a multiplicative contraction if there exists a real constant $a_3 \in [0, 1)$ such that

$d(f(x), f(y)) \leq d(x,y)^{a_3}$ for all x, y ∈ X.

Then f has a unique fixed point.

Cor.5. Putting $a_4 = a_5 = 0$, gives Ciric-contraction[3] in the sense of multiplicative metric spaces as follows:
Let f be a continuous self- mapping defined on a complete multiplicative metric space X, further f satisfies the following conditions
$d(fx, fy) \leq [d(x,fx) . d(y,fy)]^{a_1} . [d(x,fy). d(y,fx)]^{a_2} . [d(x, y)]^{a_3}$,
for all x, y ∈ X and $2a_1 + 2a_2 + a_3 < 1$ where $a_1, a_2, a_3 \in [0,1]$.
Then T has unique fixed point.

Cor.6. Putting $a_1 = a_4 = a_5 = 0$, gives Reich-contraction[9] in the sense of multiplicative metric spaces as follows:
Let f be a continuous self- mapping defined on a complete multiplicative metric space X, further f satisfies the following conditions
$d(fx, fy) \leq [d(x,fy). d(y,fx)]^{a_2} . [d(x, y)]^{a_3}$, for all x, y ∈ X and $2a_2 + a_3 < 1$ where $a_2, a_3 \in [0,1]$. Then T has unique fixed point.

Cor.7. Putting $a_1 = a_2 = a_5 = 0$gives jaggi-contraction[6] in the sense of multiplicative metric spaces as follows:

Let f be a continuous self- mapping defined on a complete multiplicative metric space X, further f satisfies the following conditions

$d(fx, fy) \leq [d(x, y)]^{a_3} . [\frac{d(x,fx)\, d(y,Ty)}{d(x,y)}]^{a_4}$,

for all x, y ∈ X and $a_3 + a_4 < 1$ where $a_3, a_4 \in [0,1]$.

Then T has unique fixed point.

Theorem2.4. Let (X, d) be a complete multiplicative metric space .Let T: X → X be almost multiplicative contraction i.e.,

$d(fx, fy) \leq \{\frac{[d(x,fx)d(y,Ty)]}{d(x,y)}\}^{\alpha} . \{d(x,y)\}^{\beta} . \{\min\{d(x,fy), d(y,fx)\}\}^{L} . \{\min\{d(x,fx), d(y,fy)\}\}^{j}$, for all x, y ∈ X where L, j ≥ 0 and α , β , γ ∈ [0,1] with α + β + j < 1.Then T has a unique fixed point in X.

Proof. Let $\{x_n\}$ be a sequence in X, defined as follows:

Let $x_0 \in X$, $f(x_0) = x_1, f(x_1) = x_2, \cdots, f(x_n) = x_{n+1}$.

If $x_n = x_{n+1}$ for some n∈ N then x_n is a fixed point of f.

Taking $x_n \neq x_{n+1}$ for all n ∈ N

Consider

$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$

$\leq \{\frac{[d(x_{n-1}, fx_{n-1})d(x_n,Tx_n)]}{d(x_{n-1},x_n)}\}^{\alpha} . \{d(x_{n-1},x_n)\}^{\beta} . \{\min\{d(x_{n-1}, fx_n), d(x_n, fx_{n-1})\}\}^{L} . \{\min\{d(x_{n-1}, fx_{n-1}), d(x_n, fx_n)\}\}^{j}$

$\leq \{\frac{[d(x_{n-1}, x_n)d(x_n, x_{n+1})]}{d(x_{n-1}, x_n)}\}^{\alpha} . \{d(x_{n-1},x_n)\}^{\beta} . \{\min\{d(x_{n-1}, x_{n+1}), d(x_n, x_n)\}\}^{L} . \{\min\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}\}^{j}$

$d(x_n, x_{n+1}) \leq \{d(x_{n+1}, x_n)\}^{\alpha}.\{d(x_{n-1},x_n)\}^{\beta}. \{\{\min\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}\}^{j}$.

Case I. When $\{\min\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n)$ then

$\{d(x_n, x_{n+1})\}^{1-\alpha} \leq \{d(x_n, x_{n-1})\}^{\beta+j}$,

$d(x_n, x_{n+1}) \leq [d(x_{n-1}, x_n)]^{h}$, (2.1)

where $h = \frac{\beta+j}{1-\alpha} < 1$.

Case II. When $\{\min\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n+1}, x_n)$ then

$\{d(x_n, x_{n+1})\}^{1-\alpha} \leq \{d(x_n, x_{n-1})\}^{\beta}. \{d(x_n, x_{n+1})\}^{j}$

$d(x_n, x_{n+1}) \leq [d(x_{n-1}, x_n)]^{h}$, (2.2)

where $h = \frac{\beta}{1-\alpha-j} < 1$, from (2.1) and (2.2) we get

,$d(x_n, x_{n+1}) \leq [d(x_{n-1}, x_n)]^{h}$.

Similarly, $d(x_{n-1}, x_n) \leq [d(x_{n-2}, x_{n-1})]^{h}$,

$d(x_n, x_{n+1}) \leq [d(x_{n-2}, x_{n-1})]^{h^2}$.

Continue like this we get,

$d(x_n, x_{n+1}) \leq [d(x_0, x_1)]^{h^n}$

For n > m, $d(x_n, x_m) \leq d(x_n, x_{n-1}) \cdot d(x_{n-1}, x_{n-2}) \cdots d(x_m, x_{m+1})$

$\leq d(x_0, x_1)^{h^{n-1}+h^{n-2}+\cdots h^m}$

$\leq d(x_0, x_1)^{\frac{h^m}{1-h}}$. This implies $d(x_n, x_m) \to 1 (n, m \to \infty)$.

Hence (x_n) is a Cauchy sequence. By the multiplicative completeness of X, there is z ∈ X such that $x_n \to z$ (n →∞).

Now we claim that u = Tu.

$d(Tu, u) \leq d(x_{n+1}, u). d(x_{n+1}, Tu)$

$\leq d(x_{n+1}, u). d(Tx_n, Tu)$

$\leq d(x_{n+1}, u). \{\frac{[d(x_n, fx_n)d(u,Tu)]}{d(x_n,u)}\}^{\alpha} . \{d(x_n, u)\}^{\beta} . \{\min\{d(x_n, fu), d(u, fx_n)\}\}^{L} . \{\min\{d(x_n, fx_n), d(u, fu)\}\}^{j}$

$\leq d(x_{n+1}, u). \{\frac{[d(x_n, x_{n+1})]}{d(x_n,u)}\}^{\alpha} . \{d(x_n, u)\}^{\beta} . \{\min\{d(x_n, u), d(u, x_{n+1})\}\}^{L} . \{\min\{d(x_n, x_{n+1}), d(u, u)\}\}^{j}$

d(Tu, u) ≤ 1,implies that d(Tu, u) = 1,Tu = u.

Hence u is fixed point of T.

Uniqueness can easily follow.

Cor.1. If L = j = 0 then we get jaggi contraction[6]in sense of multiplicative as follows:

Let (X, d) be a complete multiplicative metric space .Let T: X → X be multiplicative contraction, i.e.,

$d(fx, fy) \leq \{\frac{[d(x,fx)d(y,Ty)]}{d(x,y)}\}^{\alpha} . \{d(x,y)\}^{\beta}$, for all x, y ∈ X where α , β ∈ [0,1] with α + β < 1,then T has a unique fixed point in X.

Cor.2. If d(x, y) =1 and L = j = 0 then we get Kannan contraction[8] as follows:

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Let (X, d) be a complete multiplicative metric space. Suppose the mapping f : X → X satisfies the contraction condition

$d(fx,fy) \le (d(fx,x)\cdot d(fy,y))^{\alpha}$, for all x, y ∈ X, where $\alpha \in [0, \frac{1}{2})$.

Then f has a unique fixed point in X.

Theorem2.5. Let f be a self- mapping defined on a complete multiplicative metric space X, further f satisfies the following conditions

$$d(fx, fy) \le [d(x, y)]^{a_1}. \left[\frac{d(x,fx)\, d(y,fy)}{d(x,y).d(x,Ty).d(y,Tx)}\right]^{a_2}$$

for all x, y ∈ X and $a_1 + a_2 < 1$ where a_1, a_2 ∈[0,1] Then f has unique fixed point.

Proof. Let $\{x_n\}$ be a sequence in X, defined as follows:

Let x_0 ∈ X, $f(x_0) = x_1, f(x_1) = x_2, \cdots, f(x_n) = x_{n+1}$.

If $x_n = x_{n+1}$ for some n∈ N then x_n is a fixed point of f.

Taking $x_n \neq x_{n+1}$ for all n ∈ N

Consider

$$d(x_{n+1}, x_n) = d(fx_n, fx_{n-1})$$

$$\le [d(x_n, x_{n-1})]^{a_1}. \left[\frac{d(x_n,fx_n)\, d(x_{n-1},fx_{n-1})}{d(x_n,x_{n-1}).d(x_n,fx_{n-1}).d(x_{n-1},fx_n)}\right]^{a_2}$$

$$\le [d(x_n, x_{n-1})]^{a_1}. \left[\frac{d(x_n,x_{n+1})\, d(x_{n-1},x_n)}{d(x_n,x_{n-1}).d(x_n,x_n).d(x_{n-1},x_{n+1})}\right]^{a_2}$$

$$\le [d(x_n, x_{n-1})]^{a_1}. \left[\frac{d(x_n,x_{n+1})}{d(x_{n+1},x_{n-1})}\right]^{a_2}$$

$$[d(x_n, x_{n-1})]^{a_1}. \left[\frac{d(x_{n-1},x_{n+1})\, d(x_{n-1},x_n)}{d(x_{n+1},x_{n-1})}\right]^{a_2}$$

$d(x_{n+1}, x_n) \le [d(x_n, x_{n-1})]^{a_1+a_2}$,

$d(x_n, x_{n+1}) \le [d(x_{n-1}, x_n)]^{h}$,

where $h = a_1 + a_2 < 1$, we get $d(x_n, x_{n+1}) \le [d(x_{n-1}, x_n)]^{h}$,

Similarly, $d(x_{n-1}, x_n) \le [d(x_{n-2}, x_{n-1})]^{h}$,

$d(x_n, x_{n+1}) \le [d(x_{n-2}, x_{n-1})]^{h^2}$.

Continue like this we get,

$d(x_n, x_{n+1}) \le [d(x_0, x_1)]^{h^n}$

For n > m, $d(x_n, x_m) \le d(x_n, x_{n-1}) \cdot d(x_{n-1}, x_{n-2}) \cdots d(x_m, x_{m+1})$

$$\le d(x_0, x_1)^{h^{n-1}+h^{n-2}+\cdots h^m}$$

$\le d(x_0, x_1)^{\frac{h^m}{1-h}}$. This implies $d(x_n, x_m) \to 1$(n, m → ∞).

Hence (x_n) is a Cauchy sequence. By the multiplicative completeness of X, there is z ∈ X such that $x_n \to z$ (n →∞).

Now we claim that z = fz.

$d(fz, z) \le d(x_{n+1}, z). d(x_{n+1}, fz)$

$\le d(x_{n+1}, z). d(fx_n, fz)$

$$\le d(x_{n+1}, z). [d(x_n, z)]^{a_1}. \left[\frac{d(x_n,fx_n)\, d(z,fz)}{d(x_n,z).d(x_n,Tz).d(z,Tx_n)}\right]^{a_2}$$

$d(fz, z) \le 1$implies that fz = z. Hence z is fixed point of f.

Uniqueness can be easily found.

Cor.1. Putting a_2 = 0, gives Banach-contraction[8] in the sense of multiplicative metric spaces.

Theorem 2.6. Let (X, d) be a complete multiplicative metric space. Suppose the mapping

f : X → X be a self- mapping satisfies the condition

$$d(fx,fy) \le [d(x,y)]^{a_1}.[d(x,fy)]^{a_2}.[d(fx,y)]^{a_3}.[d(fy,y)]^{a_4}.[d(fx,x)]^{a_5},$$

for all x, y ∈ X, where $a_1, a_2, a_3, a_4, a_5 \ge 0$ and $a_1 + 2a_2 + 2a_3 + a_4 + a_5 < 1$

Then f has a unique fixed point in X.

Proof. Let$\{x_n\}$be a sequence in X, defined as follows.

Let x_0 ∈ X, $f(x_0) = x_1, f(x_1) = x_2, \cdots, f(x_n) = x_{n+1}, \cdots$.

Consider

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$$

$$\le [d(x_{n-1},x_n)]^{a_1}.[d(x_{n-1},fx_n)]^{a_2}.[d(fx_{n-1},x_n)]^{a_3}.[d(fx_n,x_n)]^{a_4}.[d(fx_{n-1},x_{n-1})]^{a_5}$$

$$\le [d(x_{n-1},x_n)]^{a_1}.[d(x_{n-1},x_{n+1})]^{a_2}.[d(x_n,x_n)]^{a_3}.[d(x_{n+1},x_n)]^{a_4}.[d(x_{n-1},x_n)]^{a_5}$$

$$\le [d(x_{n-1},x_n)]^{a_1}.[d(x_{n-1},x_{n+1})]^{a_2}.[d(x_n,x_{n+1})]^{a_2}. [d(x_{n-1},x_n)]^{a_3}.[d(x_n,x_{n+1})]^{a_3}.[d(x_{n+1},x_n)]^{a_4}.[d(x_{n-1},x_n)]^{a_5}$$

$$\le [d(x_{n-1},x_n)]^{a_1+a_2+a_3+a_5}. [d(x_{n+1},x_n)]^{a_2+a_3+a_4}$$

$d(x_n, x_{n+1}) \le [d(x_{n-1}, x_n)]^{h}$,

where $h = \frac{a_1+a_2+a_3+a_5}{1-(a_2+a_3+a_4)} < 1$.

Similarly, $d(x_{n-1}, x_n) \le [d(x_{n-2}, x_{n-1})]^{h}$,

$d(x_n, x_{n+1}) \le [d(x_{n-2}, x_{n-1})]^{h^2}$.

Continue like this we get,

$d(x_n, x_{n+1}) \le [d(x_0, x_1)]^{h^n}$

For n > m, $d(x_n, x_m) \le d(x_n, x_{n-1}) \cdot d(x_{n-1}, x_{n-2}) \cdots d(x_m, x_{m+1})$

$\le d(x_0, x_1)^{h^{n-1}+h^{n-2}+\cdots h^m}$

$\le d(x_0, x_1)^{\frac{h^m}{1-h}}$. This implies $d(x_n, x_m) \to 1$(n, m → ∞).

Hence (x_n) is a Cauchy sequence. By the multiplicative completeness of X, there is z ∈ X such that $x_n \to z$ (n →∞).

Now we show that z is fixed point of f .

d(fz, z) ≤ d(fx_n,fz). d(fx_nz)

$\leq [d(z,x_n)]^{a_1}.[d(x_n,fz)]^{a_2}.[d(fx_n,z)]^{a_3}.[d(fz,z)]^{a_4}.[d(fx_n,x_n)]^{a_5}$

d(fz, z) ≤ $[d(z,fz)]^{a_2+a_4}$ gives fz = z, i.e., z is a fixed point of f.

Uniqueness: Suppose z, w (z ≠ w) be two fixed point of f, then

d(z, w) = d(fz, fw)

$\leq [d(z,w)]^{a_1}.[d(z,fw)]^{a_2}.[d(fz,w)]^{a_3}.[d(fw,w)]^{a_4}.[d(fz,z)]^{a_5}$

d(z, w) ≤ $[d(z,w)]^{a_1+a_2+a_3}$ this implies that d(z, w) = 1 i.e., z = w.

Hence f has a unique fixed point .

Cor.1. Putting $a_2 = a_3 = a_4 = a_5 = 0$ gives Banach-contraction[8].

Cor.2. Putting $a_1 = a_2 = a_3 = 0$, $a_4 = a_5$ gives Kannan-contraction[8].

Cor.3. Putting $a_1 = a_4 = a_5 = 0$, $a_2 = a_3$ gives Chatterjea-contraction[8].

Cor.5. Putting $a_4 = a_5 = 0$, gives Isufati results[5] in the sense of multiplicative metric spaces.

Cor.7. Putting $a_4 = a_5 = 0$, $a_2 = a_3$ gives Reich results[9] in the sense of multiplicative metric spaces.

3. Application to the existence of solutions of multiplicative integral equations

Let X = C([1,T];$\mathbb{R}^+$) for sufficiently small T > 1 be the set of continuous functions defined on closed interval [1,T] and d: X×X →$\mathbb{R}^+$ be defined as d(x, y) = $\sup_{t\in[1,T]} \left|\frac{x(t)}{y(t)}\right|$ for x, y ∈ X.

Then (X, d) is complete multiplicative metric spaces.

Consider the multiplicative integral equation

x(t)=u(t).$\int_1^t (K(t,s)f(s,x(s)))^{ds}$,

(3.1)

and let F: X→X defined by

F(x)(t)=u(t).$\int_1^t (K(t,s)f(s,x(s)))^{ds}$

(3.2)

We assume that

(a) f: [1,T] → $\mathbb{R}^+$ is continuous;

(b) u: [1,T] → $\mathbb{R}^+$ is continuous;

(c) K: [1,T] × $\mathbb{R}^+$ → $\mathbb{R}^+$ is continuous.

(d) for every x, y ∈ X ,

we have $\left|\frac{f(s,\ x(s))}{f(s,\ y(s))}\right| \leq ((\lambda_1)^{\left|\frac{x(s)}{y(s)}\right|}.(\lambda_2)^{\left|\frac{x(s)}{Fy(s)}\right|}.(\lambda_3)^{\left|\frac{Fx(s)}{y(s)}\right|}.(\lambda_4)^{\left|\frac{Fy(s)}{y(s)}\right|}.(\lambda_5)^{\left|\frac{Fx(s)}{x(s)}\right|})$,

where $\lambda_i \geq 0$, i = 1 to 5 and $\lambda_1 + 2\lambda_2 + 2\lambda_3 + \lambda_4 + \lambda_5 < 1$.

(e) $|t - t_0|$ ≤ K, for K > 0 sufficiently small K^{λ_i} <1, i = 1 to 5.

Theorem 3.1. Under the assumptions (a) to (e), the integral equation (3.1) has a unique solution in X.

Proof. Consider the mappings F: X→X defined by (3.2). Notice that the existence of a solution for the multiplicative integral equation (3.1) is equivalent to the existence of a fixed point for the map F.

By condition (d) , we have $\sup_{t\in[1,T]} \left|\frac{F(x)(t)}{F(y)(t)}\right| = \sup_{t\in[1,T]} \left|\frac{u(t).\int_1^t (K(t,s)f(s,x(s)))^{ds}}{u(t).\int_1^t (K(t,s)f(s,y(s)))^{ds}}\right| = \sup_{t\in[1,T]} \left|\frac{\int_1^t (K(t,s)f(s,x(s)))^{ds}}{\int_1^t (K(t,s)f(s,y(s)))^{ds}}\right|$

$= \sup_{t\in[1,T]} \left|\frac{\int_1^t (f(s,x(s)))^{ds}}{\int_1^t (f(s,y(s)))^{ds}}\right| = \sup_{t\in[1,T]} \left(\left|\frac{\int_1^t f(s,x(s))}{\int_1^t f(s,y(s))}\right|\right)^{ds} \leq \sup_{t\in[1,T]} \left(\int_1^t \left|\frac{f(s,x(s))}{f(s,y(s))}\right|\right)^{ds}$

$\leq \sup_{t\in[1,T]} (\int_1^t ((\lambda_1)^{\left|\frac{x(s)}{y(s)}\right|}.(\lambda_2)^{\left|\frac{x(s)}{Fy(s)}\right|}.(\lambda_3)^{\left|\frac{Fx(s)}{y(s)}\right|}.(\lambda_4)^{\left|\frac{Fy(s)}{y(s)}\right|}.(\lambda_5)^{\left|\frac{Fx(s)}{x(s)}\right|}))^{ds}$ by (d)

$\leq (\int_1^t (\sup_{t\in[1,T]} (\lambda_1)^{\left|\frac{x(s)}{y(s)}\right|}.\sup_{t\in[1,T]} (\lambda_2)^{\left|\frac{x(s)}{Fy(s)}\right|}.\sup_{t\in[1,T]} (\lambda_3)^{\left|\frac{Fx(s)}{y(s)}\right|}.\sup_{t\in[1,T]} (\lambda_4)^{\left|\frac{Fy(s)}{y(s)}\right|}.\sup_{t\in[1,T]} (\lambda_5)^{\left|\frac{Fx(s)}{x(s)}\right|}))^{ds}$

$\leq (\int_1^t ((\lambda_1)^{\sup_{t\in[1,T]}\left|\frac{x(s)}{y(s)}\right|})^{ds}.(\int_1^t ((\lambda_2)^{\sup_{t\in[1,T]}\left|\frac{x(s)}{Fy(s)}\right|})^{ds}.$
$(\int_1^t ((\lambda_3)^{\sup_{t\in[1,T]}\left|\frac{Fx(s)}{y(s)}\right|})^{ds}.(\int_1^t ((\lambda_4)^{\sup_{t\in[1,T]}\left|\frac{Fy(s)}{y(s)}\right|})^{ds}.$
$(\int_1^t ((\lambda_5)^{\sup_{t\in[1,T]}\left|\frac{Fx(s)}{x(s)}\right|})^{ds}.$

$\leq (\int_1^t ((\lambda_1)^{d(x,y)})^{ds}.(\int_1^t ((\lambda_2)^{d(x,Fy)})^{ds}.$
$(\int_1^t ((\lambda_3)^{d(Fx,y)})^{ds}.(\int_1^t ((\lambda_4)^{d(Fy,y)})^{ds}.$
$(\int_1^t ((\lambda_5)^{d(Fx,x)})^{ds}$

$\leq (\int_1^t (1)^{ds})^{(\lambda_1)^{d(x,y)}}.(\int_1^t (1)^{ds})^{(\lambda_2)^{d(x,Fy)}}.$
$(\int_1^t (1)^{ds})^{(\lambda_3)^{d(Fx,y)}}.(\int_1^t (1)^{ds})^{(\lambda_4)^{d(Fy,y)}}$
$(\int_1^t (1)^{ds})^{(\lambda_5)^{d(Fx,x)}}.$

$\leq (|t-t_0|)^{(\lambda_1)^{d(x,y)}}.(|t-t_0|)^{(\lambda_2)^{d(x,Fy)}}.(|t-$
t0)(λ3)d(Fx,y).

$(|t-t_0|)^{(\lambda_4)^{d(Fy,y)}}.(|t-t_0|)^{(\lambda_5)^{d(Fx,x)}}.$
$\leq (K)^{(\lambda_1)^{d(x,y)}}.(K)^{(\lambda_2)^{d(x,Fy)}}.(K)^{(\lambda_3)^{d(Fx,y)}}.$
$(K)^{(\lambda_4)^{d(Fy,y)}}.(K)^{(\lambda_5)^{d(Fx,x)}}$, from (e)
$\leq (K^{\lambda_1})^{d(x,y)} . (K^{\lambda_2})^{d(x,Fy)}.$
$(K^{\lambda_3})^{d(Fx,y)}.(K^{\lambda_4})^{d(Fy,y)}.(K^{\lambda_5})^{d(Fx,x)}$
$\leq$
$(d(x,y))^{K^{\lambda_1}}.(d(x,Fy))^{K^{\lambda_2}}.(d(Fx,y))^{K^{\lambda_3}}.$
$(d(Fy,y))^{K^{\lambda_4}}.(d(Fx,x))^{K^{\lambda_5}}$ as

$K^{\lambda_i} < 1$, i = 1 to 5.

All conditions of the theorem 2.6 are satisfied and hence the mapping F has a unique fixed point in X = C([1,T],$\mathbb{R}^+$) of the multiplicative integral equation (3.1).

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